When Are Signals Complements or Substitutes?

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Introduction
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• We develop notions of complementarity and substitutability of two signals
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  - There is a single decision maker (DM) who has to make a decision
  - Outcome of the decision depends on an unknown state of the world
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• We consider the following situation:
  – There is a single decision maker (DM) who has to make a decision
  – Outcome of the decision depends on an unknown state of the world
  – The DM can potentially observe two signals ...

... that contain information about the state of the world
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• Question: How does the *incentive to acquire a signal* depend on ...

... whether the DM *does or does not* have the other signal already?
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• Signals are complements when ...
  ... the incentive to acquire one signal increases ...
  ... as the other signal becomes available
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• Signals are complements when ...
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• Signals are substitutes when ...
  ... the incentive to acquire one signal decreases ...
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• The incentive to acquire a signal depends on ...

... the *specific* decision problem at hand
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  ... the specific decision problem at hand

• This paper: we say that signals are compl. (subst.) when ...
  – ... they are compl. (subst.) in ALL decision problems
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- We seek **conditions on the joint signal distribution** such that ...
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- This approach is in the spirit of **Blackwell** (1951)
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- This paper: we say that signals are compl. (subst.) when ...
  - ... they are compl. (subst.) in ALL decision problems

- We seek conditions on the joint signal distribution such that ...
  ... signals are complements or substitutes

- This approach is in the spirit of Blackwell (1951)
  - many pairs of signals will be neither complements nor substitutes
Example: Complements
Example: Complements

- Two states: $s \in \{-1, +1\}$, two realizations per signal: $\sigma_i \in \{-1, +1\}$
Example: Complements

- **Two** states: $s \in \{-1, +1\}$, **two** realizations per signal: $\sigma_i \in \{-1, +1\}$
- **Joint** signal distribution conditional on the state

\[
\begin{array}{c|cc}
\sigma_1 \backslash \sigma_2 & -1 & +1 \\
\hline
-1 & 0 & 1/2 \\
+1 & 1/2 & 0 \\
\end{array}
\]

State $-1$

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- **State** \(-1\)
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- Each signal **alone** is uninformative, i.e.
  \[ P[\sigma_i \mid -1] = P[\sigma_i \mid +1] \]
- But **jointly** signals fully reveal the state
- Example: signal 1 = **coded communication**, signal 2 = **encryption code**
Contribution of the paper
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• First attempt to \textit{systematically} conceptualize compl. (subst.) of information
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• We provide a **general** characterization of compl. (subst.):
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- We provide a **general** characterization of compl. (subst.):
  
  – relate compl. (subst.) to a Blackw.-comparison of two auxiliary signals
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• **Applications**: second price auction, strategic information acquisition
Literature
Literature

• Radner and Stiglitz (1984): **Non-concavity** in the value of information
  – specific decision problem
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  – specific decision problem

• Informational complementarity in **specific contexts**
  – Milgrom and Weber (1982): Auctions
  – Persico (2004): Information acquisition in committees
Set–up and Definitions: Statistical environment
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• A finite state space $S$
Set-up and Definitions: Statistical environment

- A finite state space $S$
- Two signals, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, with realizations, $\sigma_1$ and $\sigma_2$, in the finite sets $S_1$ and $S_2$
Set–up and Definitions: Statistical environment

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• Joint distribution of $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ conditional on state $s$:

$$p_s : S_1 \times S_2 \rightarrow [0, 1]$$
Set–up and Definitions: Statistical environment

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- Joint distribution of $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ conditional on state $s$:
  \[ p_s : S_1 \times S_2 \rightarrow [0, 1] \]
- Marginal distribution of $\tilde{\sigma}_i$ conditional on state $s$:
  \[ p_{i,s} : S_i \rightarrow [0, 1] \]
Set–up and Definitions: Decision problem
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- A decision problem is a triple \((\pi, A, u)\) where
Set–up and Definitions: Decision problem

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  - \(\pi\) is a probability distribution on the state space \(S\)
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  - \(u : A \times S \rightarrow \mathbb{R}\) is a (state-dependent) utility–function
Set-up and Definitions: Value
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- **Fix** a decision problem \((\pi, A, u)\)
Set–up and Definitions: Value

- **Fix** a decision problem \((\pi, A, u)\)

- Value of signals
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V_\emptyset \equiv \max_{a \in A} \sum_{s \in S} u(a, s) \pi(s)
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\[
V_{1,2} \equiv \sum_{\sigma_1 \in S_1} \sum_{\sigma_2 \in S_2} \max_{a \in A} \sum_{s \in S} u(a, s) p_s(\sigma_1, \sigma_2) \pi(s)
\]
Complementarity and substitutability
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• Signals are complements if for all decision problems

\[ V_{1,2} - V_1 \geq V_2 - V_\emptyset \quad \text{and} \quad V_{1,2} - V_2 \geq V_1 - V_\emptyset. \]
Complementarity and substitutability

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- “WTP” for one signal goes **up** as the other becomes available.
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– “WTP” for one signal goes **up** as the other becomes available
– Inequalities are equivalent

• Signals are **substitutes** if for all decision problems

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Complementarity and substitutability

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– “WTP” for one signal goes **down** as the other becomes available
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A general result
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- Re-write complementarity inequality:
A general result

- Re-write complementarity inequality:

\[ V_{1,2} + V_{\emptyset} \]
A general result

- Re-write complementarity inequality:

\[ V_{1,2} + V_{\emptyset} \geq \]
A general result

- Re-write complementarity inequality:

\[ V_{1,2} + V_\emptyset \geq V_1 + V_2 \]
A general result

• Re-write complementarity inequality:

\[ V_{1,2} + V_{\emptyset} \geq V_1 + V_2 \iff \]
A general result

- Re-write complementarity inequality:

\[ V_{1,2} + V_{\emptyset} \geq V_1 + V_2 \iff \frac{1}{2} \cdot V_{1,2} + \frac{1}{2} \cdot V_{\emptyset} \]
A general result

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- LHS = value of a signal \( \tilde{\sigma}_L \), defined by:

  - with \( \frac{1}{2} \) get signal \((\tilde{\sigma}_1, \tilde{\sigma}_2)\) and with \( \frac{1}{2} \) get no signal
A general result

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- **LHS** = value of a signal $\tilde{\sigma}_L$, defined by:
  - with $1/2$ get signal $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ and with $1/2$ get no signal

- **RHS** = value of a signal $\tilde{\sigma}_R$, defined by:
  - with $1/2$ get signal $\tilde{\sigma}_1$ and with $1/2$ get signal $\tilde{\sigma}_2$
A general result

• Re-write complementarity inequality:

\[ V_{1,2} + V_\emptyset \geq V_1 + V_2 \iff \frac{1}{2} \cdot V_{1,2} + \frac{1}{2} \cdot V_\emptyset \geq \frac{1}{2} \cdot V_1 + \frac{1}{2} \cdot V_2 \]

• LHS = value of a signal \( \tilde{\sigma}_L \), defined by:
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• RHS = value of a signal \( \tilde{\sigma}_R \), defined by:
  – with \( \frac{1}{2} \) get signal \( \tilde{\sigma}_1 \) and with \( \frac{1}{2} \) get signal \( \tilde{\sigma}_2 \)

• Signals are complements \( \iff \) \( \tilde{\sigma}_L \) is more valuable than \( \tilde{\sigma}_R \)
A general result

- Signals are complements ⇔
  - Blackwell–dominates $\tilde{\sigma}_L$ Blackwell–dominates $\tilde{\sigma}_R$: $\tilde{\sigma}_R = \tilde{\sigma}_L + \text{noise}$
A general result

• Signals are complements ⇔
  ◊ \( \tilde{\sigma}_L \) Blackwell–dominates \( \tilde{\sigma}_R \): \( \tilde{\sigma}_R = \tilde{\sigma}_L + \text{noise} \)

• Signals are substitutes ⇔
  ◊ \( \tilde{\sigma}_R \) Blackwell–dominates \( \tilde{\sigma}_L \): \( \tilde{\sigma}_L = \tilde{\sigma}_R + \text{noise} \)
Symmetric Binary Example
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- Two states: $S = \{a, b\}$
Symmetric Binary Example

• Two states: \( S = \{ a, b \} \)

• Two realizations per signal: \( S_1 = \{ \alpha, \beta \}, \quad S_2 = \{ \hat{\alpha}, \hat{\beta} \} \)
Symmetric Binary Example

- Two states: \( S = \{a, b\} \)
- Two realizations per signal: \( S_1 = \{\alpha, \beta\}, \quad S_2 = \{\hat{\alpha}, \hat{\beta}\} \)

- Symmetry:

```
\[\begin{array}{c|cc}
\alpha & \hat{\alpha} & \hat{\beta} \\
\hline
x_a & y_a \\
\beta & y_a & z_a \\
\end{array}\]
```

```
\[\begin{array}{c|cc}
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\hline
x_b & y_b \\
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```

state \( a \)  state \( b \)
Symmetric Binary Example

- **Two states:** \( S = \{ a, b \} \)

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- **Symmetry:**

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- **Wlog:** \( x_a + y_a > x_b + y_b \) (i.e. \( \alpha \) and \( \hat{\alpha} \) indicate state \( a \))
Symmetric Binary Example

- **Two** states: \( S = \{ a, b \} \)

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  state a \quad state b

- **Wlog:** \( x_a + y_a > x_b + y_b \) (i.e. \( \alpha \) and \( \hat{\alpha} \) indicate state \( a \))

- **Assume:** \( \forall \sigma_i \exists s : p_{i,s}(\sigma_i) > 0, \ \exists (\sigma_1, \sigma_2) : p_a(\sigma_1, \sigma_2) \neq p_b(\sigma_1, \sigma_2) \)
Symmetric Binary Example: Substitutes
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• **Proposition**: Signals are substitutes \(\Leftrightarrow\)
Symmetric Binary Example: Substitutes

- **Proposition**: Signals are substitutes ⇔

  \[ p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0 \text{ for all } s \]
Symmetric Binary Example: Substitutes

• Proposition: Signals are substitutes ⇔

◊ $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for all $s$

◊ i.e. perfect correlation: observing $\sigma_i$ fully reveals $\sigma_j$
Symmetric Binary Example: Substitutes

- **Proposition**: Signals are substitutes $\iff$
  
  $\diamond \quad p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for all $s$

  $\diamond$ i.e. **perfect correlation**: observing $\sigma_i$ fully reveals $\sigma_j$

- **Proof**: “$\Leftarrow$”
Symmetric Binary Example: Substitutes

• **Proposition**: Signals are substitutes ⇔
  
  ◦ $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for all $s$

  ◦ i.e. **perfect correlation**: observing $\sigma_i$ fully reveals $\sigma_j$

• Proof: “⇐”

  – If signals are perfectly correlated: $V_{1,2} - V_i = 0$
Symmetric Binary Example: Substitutes

• **Proposition:** Signals are substitutes $\iff$

  $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for all $s$

  i.e. **perfect correlation:** observing $\sigma_i$ fully reveals $\sigma_j$
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- **Proposition:** Signals are substitutes ⇔
  
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• Proposition: Signals are substitutes ⇔

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• Proof: “⇒”

– Spose signals are not perfectly correlated.
Symmetric Binary Example: Substitutes

- **Proposition:** Signals are substitutes $\iff$
  - $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for all $s$
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- **Proof:** “$\Rightarrow$”
  - Spose signals are not perfectly correlated.
  - Then there is a two-action problem such that:
Symmetric Binary Example: Substitutes

- **Proposition**: Signals are substitutes ⇔
  
  - $p_s(\alpha, \widehat{\beta}) = p_s(\beta, \widehat{\alpha}) = 0$ for all $s$
  
  - i.e. perfect correlation: observing $\sigma_i$ fully reveals $\sigma_j$

- **Proof**: “⇒”
  
  - Spose signals are not perfectly correlated.
  
  - Then there is a two-action problem such that:
  
  - Observing $\alpha$ or observing $\beta$ does NOT induce a switch in action ...
Symmetric Binary Example: Substitutes

• Proposition: Signals are substitutes ⇔

  ◦ \[ p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0 \] for all \( s \)

  ◦ i.e. perfect correlation: observing \( \sigma_i \) fully reveals \( \sigma_j \)

• Proof: “⇒”

  – Spose signals are not perfectly correlated.

  – Then there is a two-action problem such that:

  – Observing \( \alpha \) or observing \( \beta \) does NOT induce a switch in action ...

  – ... but observing, say \( (\beta, \hat{\beta}) \) does \( \Rightarrow V_{1,2} - V_j > V_i - V_\emptyset = 0 \)
Symmetric Binary Example: Complements
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• **Proposition:** Signals are **complements** ⇔
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- **Right** condition: \(\hat{\beta}\) “reverses the meaning” of \(\beta\)
Symmetric Binary Example: Complements

- Idea of the proof
Symmetric Binary Example: Complements

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  – Step 1: It is enough to consider two-action problems only
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  – Step 1: It is enough to consider two-action problems only
  – Step 2: Calculate values in two-action problems ("straightforward")

• In step 1 we use that signals are symmetric and binary
  – does presumably not work with more than two signal realizations
Application: Second price auction
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- Two bidders, $i, j$
Application: Second price auction

- **Two** bidders, \( i, j \)

- **One object with value** \( s \in \{0, 1\} \) **common** to both bidders
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- Focus on symmetric equilibria
Second price auction
Second price auction

• For this talk: each realization \((\sigma_1, \sigma_2)\) has strictly positive probability
Second price auction

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  \[
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  3. \(\text{post}(\beta) \leq \text{post}(\alpha, \hat{\beta})\):
SPA: Case (1)

• Proposition: If \( \text{post}(\alpha) < \text{post}(\alpha, \hat{\beta}) < \text{post}(\beta) \)
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- Remark: Affiliation implies (1)
SPA: Case (2)
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- Complementarity \textbf{implies} (2) when \( \hat{\alpha} \) reverses the meaning of \( \alpha \):
  \[
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Why are MW strategies no equ?

$(\beta, \hat{\alpha})$
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- Suppose \((\text{post}(\alpha, \hat{\alpha}), \text{post}(\beta, \hat{\beta}))\) was an equilibrium
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\[\Rightarrow\] negative utility
Intuition for zero utility result

\((\beta, \hat{\alpha})\)
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- Consider a high signal bidder \((\beta)\). \(Y\) = support of his bids
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1. win at price in $Y$ $\Rightarrow$ zero utility (otherwise: incentive to deviate)

2. win at price not in $Y$ $\Rightarrow$ rival bidder is $\hat{\alpha}$

   $\Rightarrow$ value of the good is lowest possible: $\text{post}(\beta, \hat{\alpha})$
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   \(\Rightarrow\)  value of the good is lowest possible: \(post(\beta, \hat{\alpha})\)

   - Moreover, in a symmetric equ.: price \(\geq\) lowest possible value

   \(\Rightarrow\)  zero utility
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Intuition for positive utility result

- Consider a high signal bidder \((\beta)\). \(Y = \text{support of his bids}\)

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2. win at price not in \(Y\)
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Intuition for positive utility result

- Consider a high signal bidder ($\beta$). $Y =$ support of his bids

1. win at price in $Y$ $\Rightarrow$ zero utility

2. win at price not in $Y$

$\Rightarrow$ value of the good is highest possible: $\text{post}(\beta, \hat{\alpha})$

- It can be shown:
  - low signal bidder ($\hat{\alpha}$) bids outside of $Y$ with pos. prob.
  - bids outside of $Y$ are below the bids in $Y$

$\Rightarrow$ high signal bidder makes strictly pos. utility at such bids
Second price auction: summary
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• Complementarity is sufficient, but not necessary for this
Second price auction: summary

- Under complementarity:
  - there is no symmetric equilibrium in pure strategies
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- Complementarity is sufficient, but not necessary for this

- Reason: in contrast to the standard case with affiliated signals, ...
  ... signals are not well-ordered under complementarity
Conclusion and further questions
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• Strong notion
  – restrict set of decision problems
  – e.g. monotone problems a la Athey and Levin (2001)
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• Sequential information acquisition
  – what is the incentive to acquire an additional signal ...

... conditional on having observed one signal already