On Optimal Communication Networks

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Conference Complementarities and Information
Outline

1. Introduction

2. Model and Results

3. A Family of Networked Communication Processes
Example (Krackhardt and Hanson, 1993)

- The network is incomplete.
- Different people have different information.
The Model: Information Structure

**Common Prior**

Prior: \( \theta \sim N(\theta_0, \phi_\theta) \)

**Communication Process, \( P \)**

*Communication Report:* \( y^P_i = \theta + \epsilon^P_i \)

where \( \epsilon^P | \theta \sim MN \left( 0, \phi_\epsilon \Sigma^P \right) \)

We assume in this presentation that \( \sigma_{ij} \leq \sigma_{ii} \) for all \( j \).
Posterior Expectations

Linear precision weights

\[ \mathbb{E}_i^P [\theta] = (1 - f_i^P) \theta_0 + f_i^P \hat{y}_i ; \quad \mathbb{E}_i^P [y_j^P] = (1 - \omega_{ji}^P) \theta_0 + \omega_{ji}^P \hat{y}_i \]

With \( \alpha = \phi_\epsilon / \phi_\theta \),

\[ f_i^P = \frac{1}{1 + \alpha \sigma_{ii}^P} \]
\[ \omega_{ij}^P = \frac{1 + \alpha \sigma_{ij}^P}{1 + \alpha \sigma_{jj}^P} \]
The Model: Payoffs

Let $r \in (0, 1)$.

$$u_i = -(1 - r)(a_i - \theta)^2 - r \frac{1}{n-1} \sum_{j \neq i} (a_i - a_j)^2 \quad i \in \mathcal{N}$$

Decision Problem

Coordination Problem

$\Rightarrow$ beauty contest game.
Equilibrium Analysis

Best- Replies

$$BR_i(a_{-i}) = (1 - r) E^p_i[\theta] + r \frac{1}{n-1} \sum_{j \neq i} E^p_j[a_j] \quad i \in N$$

Nestedness and q-order beliefs ($\theta_0 = 0$)

$$E^p_{i_1}E^p_{i_2} \cdots E^p_{i_q}[\theta] = f^p_{i_q} \omega^p_{i_q} i_{q-1} \cdots \omega^p_{i_2 i_1} \hat{y}_{i_1}$$

$\Rightarrow$ Higher-order beliefs are also linear on the information each agent receives.
The Knowledge Index

Let $\rho = \frac{r}{n-1}$. The knowledge index of agent $i$ aggregates all levels of beliefs discounted by the common factor $\rho$. Formally,

$$k^P = (1 - r) (I - \rho \Omega)^{-1} \cdot 1 = (1 - r) \left( I + \rho \Omega + \rho^2 \Omega^2 + \cdots \right) \cdot 1$$

The knowledge index is in $[0, 1]$. It reaches one with fully informative input signals; otherwise, it varies across agents.
The unique Bayes-Nash equilibrium for the communication process $\mathbb{P}$ has linear strategies

$$a_i^\mathbb{P} = \left(1 - k_i^\mathbb{P}\right) \theta_0 + k_i^\mathbb{P} \mathbb{E}_i^\mathbb{P} [\theta]$$

- Agents rely on their signal in proportion to their knowledge index; otherwise, the mean prior acts as a focal point.
Theorem

If $\theta_0 = 0$ the equilibrium aggregate ex-ante payoffs are

$$U_P = (1 - r) \phi_\theta \left[ \sum_{i=1}^{n} f_i^P \left( k_i^P \right)^2 - n \right] \leq 0$$

- Under complete information ($\phi_\theta = 0$), no welfare loss.

$\Rightarrow$ comparative statics with respect to coordination concern, informativeness, accuracies and correlations.
Comparative Statics: Information

Remember that

\[ U^* = (1 - r) \phi_\theta \left[ \sum_{i=1}^{n} f_i^p (k_i^p)^2 - n \right] \leq 0 \]

Proposition

If \((f(\Sigma'), \Omega(\Sigma')) \geq (f(\Sigma), \Omega(\Sigma))\), then \(U^*(r, \Sigma') \geq U^*(r, \Sigma)\).
Comparative Statics: Accuracy and Covariances.

We can write $\sigma_{ij} = \gamma_{ij} \sqrt{\sigma_{ii} \sigma_{jj}}$ with $\gamma_{ij} \in [0, 1]$ for all $i \neq j$. $\Sigma$ is fully characterized by a vector $\sigma = (\sigma_{11}, \ldots, \sigma_{nn})$ and a symmetric and zero-diagonal matrix $\Gamma = [\gamma_{ij}]$.

**Proposition**

$U^*(r, \Sigma)$ is non-decreasing with $\Gamma$ and is non-increasing with $\sigma$ (for the partial order).
Comparative Statics: the role of $r$.

**Proposition**

$k$ is monotone decreasing with $r$.

When coordination concern increases agents shift weight towards the focal point.

**Proposition**

$U^*$ is monotone increasing with $r$ whenever $\sigma_{ii} \geq \hat{\sigma}$ for all $i$, for some $\hat{\sigma}$. 

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Proposition

Let \( \tilde{\rho} = \frac{2}{1+r} \rho \). The socially optimal action is

\[
a_i^S = (1 - k_i (\tilde{\rho})) \theta_0 + k_i (\tilde{\rho}) \mathbb{E}_i [\theta]
\]

We know that at equilibrium welfare increases with the coordination concern \( r \). The socially optimal action of agent \( i \) is equivalent to the equilibrium action of the game in which the coordination concern is larger: \( \tilde{\rho} = \frac{2r}{1+r} \).
We now analyze a particular kind of communication processes in which agents receive private signals that can communicate among them.

**Individual Information**

\[
\text{Prior} \quad : \quad \theta \sim N(\theta_0, \phi_\theta) \\
\text{Private} \quad : \quad x_i = \theta + \epsilon_i \quad \text{with} \quad \epsilon_i \sim N(0, \phi_\epsilon)
\]
The $\mathbb{P}^1 (g)$ Communication Process.

Network: $g$

\[ g_{ij} = g_{ji} = 1 \text{ iff } i \text{ and } j \text{ talk}; \ g_{ii} = 1; \ g_i = g_{i1} + \cdots + g_{in} \]

The Process $\mathbb{P}^1 (g)$

- agents communicate their private signals and average them into a single communication outcome:

\[ y_i^{\mathbb{P}} = \frac{1}{g_i} \sum_{j=1}^{n} g_{ij} x_j \]

- after this communication round the process stops
Let \( c_{ij}(g) = \# \{ k : g_{ik} g_{kj} = 1 \} \). This is the number of common information sources to \( i \) and \( j \).

**Correlation of Reports**

For the \( \mathbb{P}^1(g) \) communication process we obtain that

\[
\sigma_{ij}^{\mathbb{P}^1(g)} = \frac{c_{ij}(g)}{g_i g_j}
\]
Let $c_i(g) = \sum_{j=1}^{n} c_{ij}(g)$.

**Span**

If $r$ small and $\alpha$ large, then welfare is increasing with

$$S(g) = \sum_i \frac{c_i(g)}{g_i}$$

For a given fixed supply of network links, welfare is maximal for an irregular network geometry (e.g., the star among trees, etc.)
Definition

\[ \tau(g) = \text{number of closed triples in } g \]
\[ \iota(g) = \text{number of open triples in } g \]

We say that \( g' \) is a closure of \( g \) when \( g \subseteq g' \) and \( g' \) has more closed triples and less open triples.
Proposition
If \( g' \) is a closure of \( g \) then \( k^P(g') \geq k^P(g) \).

Corollary
If \( g' \) is a closure of \( g \) then \( U(g') \geq U(g) \).
The Family $P^t(g)$ of Communication Processes

Agents communicate individual reports for $t$ rounds, $t \geq 1$.

For any $1 < k \leq t$:

$$x_i^k = \frac{1}{g_i} \sum_{j=1}^{n} g_{ij} x_j^{k-1}$$

$y_i^{P^t(g)} = x_i^t$

When $t \rightarrow \infty$ the process $P^t(g)$ leads to:

- consensual beliefs
- weighted average of initial private signals (DeMarzo, Vayanos and Zwiebel, QJE 2003)

$\Rightarrow$ For a given fixed supply of network links, welfare is maximal for a regular network geometry.
A Family of Networked Communication Processes

$r = 0.5$, $\alpha = 5$ and $t = 1, \ldots, 5$

\[
\begin{align*}
1 & \rightarrow 2 \\
1 & \rightarrow 3 \\
1 & \rightarrow 4 \\
\end{align*}
\]

kite

\[
\begin{align*}
1 & \rightarrow 2 \\
1 & \rightarrow 3 \\
1 & \rightarrow 4 \\
\end{align*}
\]

wheel

Structural Polarization as in Guimerà et al. (2002)