Interim Bayesian Nash Equilibrium
on Universal Type Spaces for Supermodular Games

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Main result

Existence of pure-strategy Bayesian Nash equilibrium with:

- **interim** formulation of a Bayesian game and no common prior.
- **interim** definition of a BNE.

Assumptions: Supermodular payoffs but otherwise general:

- **Type spaces:** any.
- **Actions:** compact metric lattice.
- **Payoffs:** measurable in the types, continuous in the actions, bounded.
- **Interim beliefs:** measurable in own type.
**Players:** \( N = \{1, \ldots, n\} \), indexed by \( i \).

For each player \( i \):

1. **Type space:** \((T_i, \mathcal{F}_i)\).

2. **Interim beliefs:** \( p_i : T_i \rightarrow M_{-i} \),
   where \( M_{-i} \) is the set of probability measures on \((T_{-i}, \mathcal{F}_{-i})\).

3. **Action set:** \( A_i \).

4. **Payoff function** \( u_i : A \times T \rightarrow \mathbb{R} \).
Strategy of player $i$: measurable $\sigma_i: T_i \rightarrow A_i$.

Let $\Sigma_i$ be set of strategies (NOT equivalence classes).

**BNE in words:** Each type of each player chooses action to maximize expected utility given beliefs for that type.

For each $i$ and each $t_i$, $\sigma_i(t_i)$ is best response to $\sigma_{-i}$:

$$
\sigma_i(t_i) \in \arg \max_{a_i} \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i}) \, dp_i(t_{-i} | t_i)
$$
• Belief mappings replaced by a common prior.

• Strategies are equivalences classes.

**BNE in words**: Player chooses a strategy before observing his type in order to maximize *unconditional* expected utility.

⇒ interim optimality for *almost every* type, rather than every type.
State of the art for general games is still Milgrom and Weber (1985):

- ex ante formulation of game and BNE;
- “distributional strategies”;
- type spaces are complete separable metric spaces;
- equicontinuous payoffs;
- absolutely continuous information (close to assuming independent types).

They finesse a problem that arises when trying a straightforward fixed-point approach to Nash equilibrium of the ex ante game: In topology that is weak enough to make strategy spaces compact, the utilities over strategies are not continuous.
Results for games with order structure

**Supermodular games**

Vives (1990) and Milgrom and Roberts (1990):

- ex ante formulation of game and BNE;
- action sets are Euclidean.

**Monotone strategies**


1. ex ante formulation of game and BNE;
2. types are Euclidean cube;
3. atomless atomless prior.
4. slightly more restrictive action sets.
Who cares about interim vs. ex post

From Myerson (2002):

Harsanyi’s point here is that the type represents what the player knows at the beginning of the game, and so calculations of the player’s expected payoff before this type is learned cannot have any decision-theoretic significance in the game.

For example, if a player’s type includes a specification of his or her gender (about which some other players are uncertain), then the normal form analysis would require us to imagine the player choosing a contingent plan of what to do if male and what to do if female, maximizing the average of male and female payoffs.
From project with Xavier Vives:

“Monotone Equilibrium in Bayesian Games of Strategic Complementarities”

**Adds these assumptions for each player**

- payoff has increasing differences in own action and profile of types;
- interim beliefs are increasing in type with respect to first-order stochastic dominance.

**Obtains also these results**

- Extremal equilibria are in strategies that are increasing in type.
- Comparative statics: if we shift interim beliefs up by first-order stochastic dominance (type-by-type) then extremal equilibria increase.
Consider any interim Bayesian game with …

1. No restriction on $T_i$.

2. $A_i$ is compact* metric lattice.

3. $u_i$ is supermodular in $a_i$ and has increasing differences in $(a_i, a_{-i})$.

4. $t_i \mapsto p_i(F_{-i}|t_i)$ is measurable for $F_{-i} \in \mathcal{F}_{-i}$.

5. $u_i$ is bounded, measurable in $t$, and continuous* in $a$.

Then the game has greatest and least pure-strategy interim BNE.

*Needed? Not usually in supermodular games. For measurability here. Can be weakened?
Main steps

Step 1

Show that each player has a greatest best reply (GBR) $\bar{\beta}_i(\sigma_{-i})$, which is increasing in $\sigma_{-i}$.

Step 2

Apply a lattice fixed-point theorem to the profile of GBR mappings

$$\bar{\beta}(\sigma) = (\bar{\beta}_1(\sigma_{-1}), \ldots, \bar{\beta}_n(\sigma_{-n}))$$

(First step 2, then step 1.)
**Assumption.** $A_i$ is a compact sublattice of Euclidean space.

**Then:** $\Sigma_i$ (set of equivalence classes) is a complete lattice.

**So we can apply Tarski’s fixed-point theorem** to $\bar{\beta} : \Sigma \rightarrow \Sigma$:

(Tarski) Suppose that $X$ is a complete lattice and that $f : X \rightarrow X$ is
an increasing function. Then $f$ has a fixed point.
$\Sigma_i$ is set of functions, not equivalence classes.

Consider partial order

$$\sigma_i' \geq \sigma_i \iff \sigma_i'(t_i) \geq \sigma_i(t_i) \ \forall t_i$$

$A_{Ti}^i$ (set of ALL functions $T_i \to A_i$) is complete lattice.

And $\Sigma_i$ is a sublattice of $A_{Ti}^i$.

But $\Sigma_i$ is not complete (typically): pointwise sup of an uncountable set of measurable functions may not be measurable.

Example: Suppose $G_i \subset T_i$ is not measurable but all singletons are measurable. Then $\{1_{\{t_i\}} \mid t_i \in G_i\} \subset \Sigma_i$ has no supremum in $\Sigma_i$. 
Definition. A partially ordered set \((X, \geq)\) is **sequentially complete** if every increasing sequence has a least upper bound and every decreasing sequence has a greatest lower bound.

Definition. Suppose \((X, \geq)\) is sequentially complete. A functional \(f : X \to \mathbb{R}\) is **sequentially order continuous** if, for every increasing or decreasing sequence \(\{x_1, x_2, \ldots\}\), \(\lim f(x_n) = f(\lim x_n)\).

Theorem. Suppose that \((X, \geq)\) is a sequentially complete, partially ordered set with a greatest element and that \(f : X \to X\) is increasing and sequentially order continuous. Then \(f\) has a greatest fixed point.

This works because \(\Sigma_i\) is sequentially complete.
How does this fixed-point theorem work?

It is just “packaged” Cournot tatônnement, as used by Vives (1990).

Proof:

• Let \( x_0 \) be greatest element of \( X \).

• For \( k \geq 1 \), define \( x_k = f(x_{k-1}) \).

• Then \( \{x_k\} \) is a decreasing sequence, …

• which converges to the greatest fixed point.
Easy:

- $\Sigma_i$ are complete lattices.
- Induced ex ante utility functions are continuous.
- Apply “optimization on complete lattices” (e.g., Milgrom and Roberts (1990)).
Easy part …

… that $\bar{\beta}_i$ is an increasing function

Follows straight from the complementarity assumptions.

Also pretty easy …

… that $\bar{\beta}_i$ is sequentially order continuous.

From continuity of $u_i$ in actions and a dominated convergence argument.

Hard part …

… that $\bar{\beta}_i$ is well-defined.

Measurability problems.
Fix $\sigma_{-i}$. (So we can suppress it as an argument.)

Objective function $\pi_i : A_i \times T_i \rightarrow \mathbb{R}$:

$$\pi_i(a_i, t_i) := \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i}) \, dp_i(t_{-i} \mid t_i).$$

Solution correspondence $\phi_i : T_i \rightarrow A_i$:

$$\phi_i(t_i) := \arg \max_{a_i \in A_i} \pi_i(a_i, t_i).$$

Greatest solution $\bar{\sigma}_i : T_i \rightarrow A_i$:

$$\bar{\sigma}_i(t_i) := \max \phi_i(t_i).$$

Is $\bar{\sigma}_i(t_i)$ well defined for all $t_i$? (Yes, optimization on lattices …)

Is $\sigma_i : T_i \rightarrow A_i$ measurable ?? (If yes, then $\bar{\sigma}_i$ is the GBR.)
Step 1. \( \pi_i \) is continuous and supermodular in \( a_i \) and bounded.

Step 2. \( \pi_i \) is measurable in \( t_i \).
Step 2: $\pi_i$ is measurable in $t_i$

Fix $a_i \in A_i$. Define

$$U_i(t_i, t_{-i}) : = u_i(a_i, \sigma_{-i}(t_i), t_i, t_{-i})$$

Then

$$\pi_i(a_i, t_i) = \int_{T-1} U_i(t_i, t_{-i}) \, dp_i(t_{-i} \mid t_i) .$$

When is

$$t_i \to \int_{T-1} U_i(t_i, t_{-i}) \, dp_i(t_{-i} \mid t_i)$$

measurable?
Abstract version

1. \((X, \mathcal{F})\) and \((Y, \mathcal{G})\) are measurable spaces;

2. \(\mathcal{M}\) is the set of probability measures on \((Y, \mathcal{G})\);

3. \(p : X \to \mathcal{M}\);

4. \(U : X \times Y \to \mathbb{R}\);

5. \(\pi(x) := \int_Y U(x, y) \, dp(y \mid x)\).

When is \(\pi : X \to \mathbb{R}\) \(\mathcal{F}\)-measurable?

Answer

- \(U : X \times Y \to \mathbb{R}\) is \(\mathcal{F} \otimes \mathcal{G}\)-measurable and bounded;

- For \(G \in \mathcal{G}\), \(x \mapsto p(G \mid x)\) is \(\mathcal{F}\)-measurable.

(Generalizes a result by Ely and Peski (2004).)
So our problem reduces to …

[Suppress subscript \(i\): \(\pi(a, t), \phi(t), \bar{\sigma}(t)\).]

Given \(\pi: A \times T \rightarrow \mathbb{R}\), that is

- continuous in \(a\);
- measurable in \(t\);
- supermodular in \(a\)

\(\pi\) is a Carathéodory function

When is \(t \mapsto \max \{\arg \max_{a \in A} \pi(a, t)\}\) measurable?
**Definition.** A correspondence \( \zeta : Y \to X \) from a measurable space \((Y, \mathcal{G})\) to a topological space \(X\) is \(\mathcal{G}\)-measurable if

\[
\zeta^w(D) := \{ y \in Y \mid \zeta(y) \cap D \neq \emptyset \} \in \mathcal{G}
\]

for every closed \(D \subset X\).

This is stronger than “\(\text{gr}(\zeta)\) is measurable”.

**Theorem.** [Castaing & Valadier] Let \((X, \mathcal{F})\) be a measurable space and let \(Y\) be a complete separable metric space. Let \(\zeta : X \to Y\) be a measurable correspondence with non-empty and closed values. Then there is a countable family \(\{f_k \mid k \in \mathbb{N}\}\) of measurable selections of \(\zeta\) such that \(\zeta(x) = \text{cl}\{f_k(x) \mid k \in \mathbb{N}\}\) for all \(x \in X\).
Show that solution correspondence \( \phi : T \rightarrow A \) is measurable.

(From Measurable Maximum Theorem)

Let \( \{\sigma_k \mid k \in \mathbb{N}\} \) be the countable collection of measurable selections.

Define recursively \( \bar{\sigma}_k(t) = \sup\{\sigma_k(t), \bar{\sigma}_{k-1}(t)\} \).

Each \( \bar{\sigma}_k \) is measurable because lattice operation \( \sup(\cdot, \cdot) \) is measurable.

\( \{\bar{\sigma}_k\} \) is increasing sequence of measurable functions; converges pointwise to measurable function.

Can show that limit is \( \bar{\sigma} \).
Add:

- complementarities between action and types;
- interim beliefs are increasing in type with respect to FOSD.

Then greatest best reply to monotone-in-type strategies is monotone in type.

Cournot tatônnement, starting at the greatest strategy profile and using greatest best replies, starts with monotone strategies, stays with monotone strategies, and converges to monotone strategies.