Working Paper

THE VALUE OF TAX SHIELDS WITH A FIXED BOOK-VALUE LEVERAGE RATIO

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Abstract

The value of tax shields depends only on the nature of the stochastic process of the net increases of debt. The value of tax shields in a world with no leverage cost is the tax rate times the current debt plus the present value of the net increases of debt. We develop valuation formulae for a company that maintains a fixed book-value leverage ratio and show that it is more realistic than to assume, as Miles-Ezzell (1980) do, a fixed market-value leverage ratio. We also show that Miles-Ezzell assume that the increase of debt is proportional to the increase of the free cash flows.

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There is no consensus in the existing literature regarding the correct way to compute the value of tax shields. Most authors think of calculating the value of the tax shield in terms of the appropriate present value of the tax savings due to interest payments on debt, but Modigliani-Miller (1963) propose to discount the tax savings at the risk-free rate[1], whereas Harris and Pringle (1985) propose discounting these tax savings at the cost of capital for the unlevered firm. Miles and Ezzel (1985) propose discounting these tax savings the first year at the cost of debt and the following years at the cost of capital for the unlevered firm. Reflecting this lack of consensus, Copeland et al. (2000, p. 482) claim that "the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct."

We show that the value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. More specifically, we prove that the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the value of the future net increases of debt.

By applying this formula to specific situations, we show that the Modigliani-Miller (1963) formula should be used when the company has a preset amount of debt; Fernández (2004), when the company expects the increases of debt to be as risky as the free cash flows (for example, if the company wants to maintain a fixed book-value leverage ratio); and Miles-Ezzell (1980), only if debt will be always a multiple of the equity market value ($D_t = L \cdot S_t$). We will argue that although $D_t = L \cdot S_t$ provides a computationally elegant solution, it is not a realistic one. What is more, we have not seen any company that follows this financing policy.

It makes much more sense to characterize the debt policy of a company with expected constant leverage ratio as a fixed book-value leverage ratio rather than as a fixed market-value leverage ratio because

1. the debt does not depend on the movements of the stock market,
2. it is easier to follow for unlisted companies, and
3. managers should prefer it because the value of tax shields is higher.

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[1] Myers (1974) proposes to discount it at the cost of debt (Kd).

Although Cooper and Nyborg (2006) disagree, this paper shows that Fernández's (2004) formula (28) (VTS = PV[Ku; D·T·Ku]) is valid, but only under the assumption that the increases of debt are as risky as the free cash flows.

The paper is organized as follows. In section 1 we derive the general formula for the value of tax shields. In section 2 we apply this formula to specific situations. Section 3 discusses the opinions of other authors. In section 4 we calculate the value of taxes for the levered and the unlevered firm. Section 5 is a numerical example. Section 6 presents the valuation formulae for finite horizons. Section 7 discusses the influence of growth on the risk of the cash flows. Section 8 concludes.

To avoid arguments about the appropriate discount rates, we will use pricing kernels. The price of an asset that pays a random amount x_t at time t is the sum of the expectation of the product of x_t and M_t, the pricing kernel for time t cash flows:

$$P_x = \sum_1^\infty E[M_t \cdot x_t] \tag{1}$$

1. General expression of the value of tax shields

The value of the debt today (D_0) is the value today of the future stream of interest minus the value today of the future stream of the increases of debt (ΔD_t):

$$D_0 = \sum_1^\infty E[M_t \cdot \text{Interest}_t] - \sum_1^\infty E[M_t \cdot \Delta D_t] \tag{2}$$

As the value of tax shields is the value of the interest times the tax rate,

$$VTS_0 = T\sum_1^\infty E[M_t \cdot \text{Interest}_t] = T \cdot D_0 + T\sum_1^\infty E[M_t \cdot \Delta D_t] \tag{3}$$

Equation (3) is valid for perpetuities and for companies with any pattern of growth. More importantly, this equation shows that the value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. The problem of equation (3) is how to calculate the value today of the increases of debt.

The value today of the levered company (V_{L0}) is equal to the value of debt (D_0) plus the value of the equity (S_0). It is also equal to the value of the unlevered company (Vu_0)[2] plus the value of tax shields due to interest payments (VTS_0):

$$V_{L0} = S_0 + D_0 = Vu_0 + VTS_0 \tag{4}$$

In the literature, the value of tax shields defines the increase in the company's value as a result of the tax saving obtained by the payment of interest.

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[2] According to our notation, $Vu_0 = \sum_1^\infty E[M_t \cdot FCF_t]$ and $S_0 = \sum_1^\infty E[M_t \cdot ECF_t]$, where FCF_t is the free cash flow of period t, and ECF_t is the equity cash flow of period t.

2. Value of the increases of debt and value of tax shields in specific situations

We apply the result in (3) to specific situations and show how this formula is consistent with previous formulae under restrictive scenarios. We will assume that

$$FCF_{t+1} = FCF_t (1 + g)(1 + \varepsilon_{t+1}) \tag{5}$$

ε_{t+1} is a random variable with expected value equal to zero ($E_t[\varepsilon_{t+1}] = 0$), but with a value today smaller than zero:

$$E_t\left[M_{t,t+1} \cdot \varepsilon_{t+1}\right] = -\frac{d}{1 + R_F} \tag{6}$$

The risk free rate corresponds to the following equation:

$$\frac{1}{1 + R_F} = \sum_{1}^{\infty} E[M_t \cdot 1] \tag{7}$$

First, we deduct the value of the unlevered equity. If $M_{t,t+1}$ is the one-period pricing kernel at time t for cash flows at time t+1,

$$Vu_t = E_t\left[M_{t,t+1} \cdot FCF_{t+1}\right] + E_t\left[M_{t,t+1} \cdot Vu_{t+1}\right] \tag{8}$$

A solution must be $Vu_t = a \cdot FCF_t$; then:

$$Vu_t = E_t\left[M_{t,t+1} \cdot FCF_{t+1}\right] + E_t\left[M_{t,t+1} \cdot aFCF_{t+1}\right] = (1+a)E_t\left[M_{t,t+1} \cdot FCF_{t+1}\right] \tag{9}$$

According to (5):

$$E_t\left[M_{t,t+1} \cdot FCF_{t+1}\right] = E_t\left[M_{t,t+1} \cdot FCF_t (1+g)\right] + E_t\left[M_{t,t+1} \cdot FCF_t (1+g)\varepsilon_{t+1}\right] \tag{10}$$

Using equation (6) and defining $Ku = (R_F + d) / (1 - d)$:

$$E_t\left[M_{t,t+1} \cdot FCF_{t+1}\right] = \frac{FCF_t(1+g)}{1+R_F} - \frac{FCF_t(1+g)d}{1+R_F} = \frac{FCF_t(1+g)(1-d)}{1+R_F} = \frac{FCF_t(1+g)}{1+Ku} \tag{11}$$

$$Vu_t = aFCF_t = (1+a)\frac{FCF_t(1+g)}{1+Ku} \tag{12}$$

$$a = \frac{(1+g)}{Ku - g} \tag{13}$$

Then:

$$Vu_t = \sum_{1}^{\infty} E[M_t \cdot FCF_t] = \frac{(1+g)}{Ku - g} FCF_t \tag{14}$$

2.1. Debt is proportional to the equity book value

If $D_t = K \cdot Ebv_t$, where Ebv is the book value of equity, then $\Delta D_t = K \cdot \Delta Ebv_t$. The increase of the book value of equity is equal to the profit after tax (PAT) minus the equity cash flow (ECF). The relationship between the profit after tax of the levered company (PAT_L) and the equity cash flow (ECF) is:

$$ECF_t = PAT_{Lt} - \Delta A_t + \Delta D_t \tag{15}$$

Notation being, ΔA_t = Increase of net assets in period t (Increase of Working Capital Requirements plus Increase of Net Fixed Assets); $\Delta D_t = D_t - D_{t-1}$ = Increase of Debt in period t.

Similarly, the relationship between the profit after tax of the unlevered company (PATu) and the free cash flow (FCF) is:

$$FCF_t = PATu_t - \Delta A_t \tag{16}$$

According to equation (15)

$$\Delta Ebv_t = PAT_{Lt} - ECF_t = \Delta A_t - \Delta D_t = \Delta D_t / K \tag{17}$$

In this situation, the increase of debt is proportional to the increases of net assets and the risk of the increases of debt is equal to the risk of the increases of assets:

$$\Delta D_t = \Delta A_t / (1+1/K) \tag{18}$$

The value today of the increases of debt is:

$$E_0[M_{0,t} \cdot \Delta D_t] = \left(\frac{K}{1+K}\right) E_0[M_{0,t} \cdot \Delta A_t] \tag{19}$$

We will assume that the increase of net assets follows the stochastic process defined by $\Delta A_{t+1} = \Delta A_t (1+g)(1+\phi_{t+1})$. ϕ_{t+1} is a random variable with expected value equal to zero $(E_t[\phi_{t+1}] = 0)$, but with a value today smaller than zero:

$$E_t[M_{t,t+1} \cdot \phi_{t+1}] = -\frac{f}{1+R_F} \tag{20}$$

Then, in the case of a growing perpetuity:

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta A_0 \left(\frac{K}{1+K}\right) \frac{(1+g)^t (1-f)^t}{(1+R_F)^t} \tag{21}$$

If we call $(1+\alpha) = (1+R_F) / (1-f)$, then

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta D_0 \frac{(1+g)^t}{(1+\alpha)^t} \tag{22}$$

α is the appropriate discount rate for the expected increases of debt. $\sum_1^\infty E[M_t \Delta D_t]$ is the sum of a geometric progression with growth rate = $(1+g)/(1+\alpha)$.

Then:

$$\sum_1^\infty E[M_t \Delta D_t] = \frac{\Delta D_0}{1 - \frac{1+g}{1+\alpha}} \frac{(1+g)}{(1+\alpha)} = \frac{\Delta D_0 (1+g)}{\alpha - g} = \frac{g D_0}{\alpha - g} \tag{23}$$

Substituting (23) in (3), we get:

$$VTS_0 = \frac{D_0 \alpha T}{(\alpha - g)} \tag{24}$$

2.2. Debt is proportional to the Equity book value and the increase of assets is proportional to the free cash flow

In this situation, $\Delta A_{t+1} = Z \cdot FCF_t$, and equation (19) is:

$$E_0[M_{0,t} \cdot \Delta D_t] = \left(\frac{K}{1+K}\right) E_0[M_{0,t} \cdot \Delta A_t] = \left(\frac{K}{1+K}\right) Z \cdot E_0[M_{0,t} \cdot FCF_t] \tag{25}$$

This is equivalent to assuming that $\phi_{t+1} = \varepsilon_{t+1}$. Then $f = d$, and $\alpha = Ku$. According to equation (14):

$$\sum_1^\infty E[M_t \Delta D_t] = \left(\frac{K}{1+K}\right) Z \cdot \sum_1^\infty E[M_t \cdot FCF_t] = \left(\frac{K}{1+K}\right) Z \frac{(1+g)FCF_0}{Ku - g} = \frac{g \cdot D_0}{Ku - g} \tag{26}$$

Substituting (26) in (3), we get:

$$VTS_0 = \frac{D_0 Ku T}{(Ku - g)} \tag{27}$$

If we assume that the increases of debt are as risky as the free cash flows ($\alpha = Ku$), the correct discount rate for the expected increases of debt is Ku, the required return to the unlevered company. (27) is equal to equation (28) in Fernández (2004).[3]

2.3. The company has a preset amount of debt

In this situation, ΔD_t is known with certainty today.

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta D_0 \frac{(1+g)^t}{(1+R_F)^t} \tag{28}$$

$\sum_1^\infty E[M_t \Delta D_t]$ is the sum of a geometric progression with growth rate $(1+g)/(1+R_F)$.

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[3] Fernández (2004) neglected to include in Equations (5) to (14) terms with expected value equal to zero. And he wrongly considered as being zero the present value of a variable with expected value equal to zero. Due to these errors, Equations (5) to (17), Tables 3 and 4, and Figure 1 of Fernández (2004) are correct only if $PV_0[\Delta A_t] = PV_0[\Delta D_t] = 0$.

Then:

$$\sum_1^\infty E[M_t \Delta D_t] = \frac{\Delta D_0}{1 - \dfrac{1+g}{1+R_F}} \frac{(1+g)}{(1+R_F)} = \frac{\Delta D_0 (1+g)}{R_F - g} = \frac{g D_0}{R_F - g} \tag{29}$$

Substituting (29) in (3), we get:

$$VTS_0 = \frac{D_0 R_F \, T}{(R_F - g)} \tag{30}$$

In this case, Modigliani-Miller (1963) applies: the appropriate discount rate for the ΔD_t (known with certainty today) is R_F, the risk-free rate. Note that, in the case of a growing perpetuity, Modigliani-Miller may be viewed as just one extreme case of section 2.2, in which $\alpha = R_F$.

2.4. Debt is proportional to the Equity market value

This is the assumption made by Miles and Ezzell (1980) and Arzac and Glosten (2005). If $D_t = L \cdot S_t$,

$$V_{Lt} = S_t + D_t = \frac{D_t}{L} + D_t = \frac{1+L}{L} D_t \tag{31}$$

We prove in Appendix 1 that the value today of the increase of debt in period 1, if the debt grows at a constant rate g, is:

$$E_0[M_{0,1} \cdot \Delta D_1] = \frac{D_0 (1+g)}{1 + Ku} - \frac{D_0}{1 + R_F} \tag{32}$$

We also prove that for $t > 1$:

$$E_0[M_{0,t} \cdot \Delta D_t] = D_0 \frac{(1+g)^{t-1}}{(1+Ku)^{t-1}} \left(\frac{(1+g)}{(1+Ku)} - \frac{1}{(1+R_F)} \right) \tag{33}$$

$\sum_1^\infty E[M_t \Delta D_t]$ is the sum of a geometric progression with growth rate $(1+g)/(1+Ku)$.

Then:

$$\sum_1^\infty E[M_t \Delta D_t] = \frac{D_0}{Ku - g} \left(g - \frac{Ku - R_F}{1 + R_F} \right) \tag{34}$$

Note that, under Miles-Ezzell $\sum_1^\infty E[M_t \Delta D_t] < 0$ if $g < \dfrac{Ku - R_F}{1 + R_F}$

Substituting (34) in (3), we get:

$$VTS_0 = \frac{D_0 R_F \, T}{(Ku - g)} \frac{(1 + Ku)}{(1 + R_F)} \tag{35}$$

It makes more sense to characterize the debt policy of a growing company with expected constant leverage ratio as a fixed book-value leverage ratio instead of as a fixed market-value leverage ratio because:

1. the debt does not depend on the movements of the stock market,
2. it is easier to follow for non-quoted companies, and
3. managers should prefer it because the value of tax shields is higher: (35) is smaller than (27) and than (24).

To assume $D_t = L \cdot E_t$ is not a good description of the debt policy of any company because if a company has only two possible states of nature in the following period, it is clear that under the worst state (low share price) the leveraged company will have to raise new equity and repay debt, and this is not the moment companies prefer to raise equity. Under the good state, the company will have to take a lot of debt and pay big dividends.

The Miles-Ezzell setup works as if the company pays all the debt (D_{t-1}) at the end of every period t and simultaneously raises all new debt D_t. The risk of raising the new debt is similar to the risk of the free cash flow and, hence, the appropriate discount rate for the expected value of the new debt is Ku.

$D_t = L \cdot S_t$ may be a computationally elegant solution (as shown in Arzac-Glosten, 2005), but unfortunately not a realistic one. Furthermore, we have not seen any company that follows this financing policy.

In Appendix 1 we also prove that if $D_t = L \cdot S_t$, then the increase of debt is proportional to the increase of the free cash flows.

3. Value of net increases of debt implied by other authors

Table I summarizes the implications of several approaches for the value of tax shields and for the value of the future increases of debt.

As we have already argued, Modigliani-Miller (1963) should be used when the company has a preset amount of debt; Fernández (2004), when we expect the increases of debt to be as risky as the free cash flow (for example, if the company wants to maintain a fixed book-value leverage ratio); and Miles-Ezzell (1980), only if debt will be a multiple of the equity market value $D_t = L \cdot E_t$. If the company maintains a fixed book-value leverage ratio and the risk of the increases of assets is different than the risk of the free cash flow, then the formulas of section 2.3 (and Appendix 2) should be applied.

Fieten et al. (2005) argue that the Modigliani-Miller formula may be applied to all situations. We have shown that it is valid only when the company has a preset amount of debt.

Cooper and Nyborg (2006) affirm that equation (27) violates value-additivity. It does not because equation (4) holds. They use only the cost of debt (R_F) or the cost of the unlevered equity (Ku) to discount the expected value of tax shields. We have seen that there are also other debt policies, for example, when the firm wants to maintain a fixed book-value leverage ratio.

4. Value today of the expected taxes

If leverage costs do not exist, then Eq. (4) could be stated as follows:

$$Vu_0 + Gu_0 = S_0 + D_0 + G_{L0} \tag{36}$$

where Gu_0 is the value today of the taxes paid by the unlevered company and G_{L0} is the value today of the taxes paid by the levered company. Eq. (36) means that the total value of the unlevered company (left-hand side of the equation) is equal to the total value of the levered company (right-hand side of the equation). Total value is the enterprise value (often called the value of the firm) plus the value today of taxes. Eq. (36) assumes that expected free cash flows are independent of leverage[4]. From (4) and (36), it is clear that the value of tax shields (VTS) is

$$VTS_0 = Gu_0 - G_{L0} \tag{37}$$

The taxes paid every year by the unlevered company ($Taxes_U$) are

$$Taxes_{Ut} = [T/(1-T)] \, PATu = [T/(1-T)] \; (FCF_t + \Delta A_t) \tag{38}$$

For the levered company, taking into consideration Eq. (17), the taxes paid each year ($Taxes_L$) are:

$$Taxes_{Lt} = [T/(1-T)] \; (ECF_t + \Delta A_t - \Delta D_t) \tag{39}$$

The present values at t=0 of equations (38) and (39) are:

$$Gu_0 = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot FCF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t]\right) = \left(\frac{T}{1-T}\right)\left(Vu_0 + \sum_1^\infty E[M_t \cdot \Delta A_t]\right) \tag{40}$$

$$G_{L0} = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot ECF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t] - \sum_1^\infty E[M_t \cdot \Delta D_t]\right) \tag{41}$$

The value of tax shields is the difference between Gu (40) and G_L (41).

5. A numerical example and a closer look at the discount rates

Appendixes 1, 2 and 3 derive additional formulae for the three theories discussed in this paper applied to growing perpetuities. Table II is a summary of the main formulae. Table III contains the main valuation results for a constant growing company. It is interesting to note that, according to Miles-Ezzell, the value today of the increases of debt is negative. According to Modigliani-Miller, Ke < Ku. It is interesting to note that while two theories assume a constant rate for the increases of debt (Modigliani-Miller assumes R_F and Fernández assumes Ku), Miles-Ezzell assumes one rate for $t = 1$ and Ku for $t>1$. The appropriate discount rate for the increase of debt in $t = 1$ is, according to Miles-Ezzell, equation (A1.9):

$$1 + K_{\Delta D1} = \frac{g(1+Ku)(1+R_F)}{g(1+R_F) + R_F - Ku}$$

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[4] When leverage costs do exist, the total value of the levered company is lower than the total value of the unlevered company. A world with leverage cost is characterized by the following relation:

$$Vu + Gu = S + D + GL + \text{Leverage Cost} > S + D + GL$$

Leverage cost is the reduction in the company's value due to the use of debt.

In our example, $K_{\Delta D1} = -280.9\%$.

Table IV contains the value of the tax shields (VTS) according to the different theories as a function of g and α. The results change dramatically when g increases. It may be seen that Modigliani-Miller is equivalent to a constant book-value leverage ratio ($D_t = L \cdot Ebv_t$), when $\alpha = R_F = 4\%$. The VTS according to M-M is infinite when $g > R_F$. Fernández (2004) is equivalent to $D_t = L \cdot Ebv_t$ when $\alpha = Ku = 9\%$.

Table V contains the value today of the increases of debt in different periods and the sum of all of them. According to Miles-Ezzell, the value today of the increases of debt in every period is negative.

We also prove that although the equity value of a growing perpetuity can be computed by discounting the expected value of the equity cash flow with a single rate Ke, the appropriate discount rates for the expected values of the equity cash flows are not constant. Table VI presents the appropriate discount rates for the expected values of the equity cash flows of our example. According to Miles-Ezzell, Ke_t is 75.77% for $t = 1$ and 9% for the rest of the periods. According to Modigliani-Miller, $Ke_t < Ku = 9\%$.

We also derive the appropriate discount rates for the expected values of the taxes. If we assume that the appropriate discount rate for the increases of assets is Ku, then the appropriate discount rate for the expected value of the taxes of the unlevered company is also Ku. But the appropriate discount rate for the expected value of the taxes of the levered company (K_{TAXL}) is different according to the three theories. Table VII presents the appropriate discount rates for the expected values of the taxes in the initial periods for our example. According to Miles-Ezzell, K_{TAXLt} is 9.79% for $t = 1$ and 9% for the rest of the periods. According to the other theories, K_{TAXLt} is higher than Ku (9%) and grows with t.

According to Modigliani-Miller and according to Fernández, the taxes of the levered company are riskier than the taxes of the unlevered company. However, according to Miles-Ezzell, both taxes are equally risky for $t > 1$.[5]

6. Valuation formulae for finite horizons

We have developed formulae for perpetuities. In this section, we show the main valuation formulae for a growing company that will produce cash flows only until period t=T. Today is t=0.

In the case of a company that maintains a fixed book-value leverage ratio, the value today of the increases of debt and the value of tax shields are[6]:

$$\sum_{1}^{T} E[M_t \Delta D_t] = \frac{gD_0}{\alpha - g}\left[1 - \left(\frac{1+g}{1+\alpha}\right)^T\right] \tag{23T}$$

$$VTS_0 = \frac{D_0 T}{(\alpha - g)}\left[\alpha - g\left(\frac{1+g}{1+\alpha}\right)^T\right] \tag{24T}$$

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[5] If the risk of the increase of assets is smaller than the risk of the free cash flows, then Miles-Ezzell provides a surprising result: the taxes of the levered company are less risky than the taxes of the unlevered company.
[6] The numbers of the formulae in this section are the same as in previous sections: we merely add a T.

If the company maintains a fixed book-value leverage ratio, and the increases of assets are as risky as the free cash flows, the value today of the increases of debt and the value of tax shields are:

$$\sum_1^T E[M_t \Delta D_t] = \frac{g \cdot D_0}{Ku - g}\left[1 - \left(\frac{1+g}{1+Ku}\right)^T\right] \tag{26T}$$

$$VTS_0 = \frac{D_0 \, T}{(Ku - g)}\left[Ku - g\left(\frac{1+g}{1+Ku}\right)^T\right] \tag{27T}$$

If we assume that the increases of debt are riskless, the value today of the increases of debt and the value of tax shields are:

$$\sum_1^T E[M_t \Delta D_t] = \frac{gD_0}{R_F - g}\left[1 - \left(\frac{1+g}{1+R_F}\right)^T\right] \tag{29T}$$

$$VTS_0 = \frac{D_0 \, T}{(R_F - g)}\left[R_F - g\left(\frac{1+g}{1+R_F}\right)^T\right] \tag{30T}$$

Under Miles-Ezzell:

$$\sum_1^T E[M_t \Delta D_t] = \frac{D_0}{Ku - g}\left(g - \frac{Ku - R_F}{1 + R_F}\right)\left[1 - \left(\frac{1+g}{1+Ku}\right)^T\right] \tag{34T}$$

$$VTS_0 = \frac{D_0 \, T}{(Ku - g)(1 + R_F)}\left(R_F(1 + Ku) - [Ku - R_F - g(1 + R_F)]\left(\frac{1+g}{1+Ku}\right)^T\right) \tag{35T}$$

7. Is Ku independent of growth?

Up to now we have assumed that Ku is constant, independent of growth. From equation (6) we know that $FCF_t = PATu_t - \Delta A_t$.

If we consider that the risk of the unlevered profit after tax (PATu) is independent of growth, and that K_{PATu} is the required return to the expected PATu, the present value of equation (6) is:

$$Vu_0 = \frac{(1+g)FCF_0}{(Ku - g)} = \frac{(1+g)PATu_0}{(K_{PATu} - g)} - \frac{gA_0}{(\alpha - g)}$$

$$Ku = g + \frac{(1+g)FCF_0}{\dfrac{(1+g)PATu_0}{(K_{PATu} - g)} - \dfrac{gA_0}{(\alpha - g)}}$$

Table VIII contains the required return to the free cash flows (Ku) as a function of α (required return to the increase of assets) and g (expected growth). It may be seen that Ku is increasing in g[7] if $\alpha < K_{PATu}$, and decreasing in g if $\alpha > K_{PATu}$

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[7] This result contradicts Cooper and Nyborg (2006), who maintain that "Ku is decreasing in g".

8. Conclusions

The value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. More specifically, the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the value today of the net increases of debt. This expression is equivalent to the difference between the present values of two different cash flows, each with its own risk: the value today of taxes for the unlevered company and the value today of taxes for the levered company. The critical parameter for calculating the value of tax shields is the value today of the net increases of debt. It may vary for different companies, but it may be calculated in specific circumstances.

For perpetual debt, the value of tax shields is equal to the tax rate times the value of debt. When the debt level is fixed, Modigliani-Miller (1963) applies, and the value of tax shields is the value today of the tax shields, discounted at the required return to debt. If the leverage ratio (D/E) is fixed at market value, then Miles-Ezzell (1980) applies with the caveats discussed. If the leverage ratio is fixed at book values and the increases of assets are as risky as the free cash flows (the increases of debt are as risky as the free cash flows), then Fernández (2004) applies. We have developed new formulas for the situation in which the leverage ratio is fixed at book values but the increases of assets have a different risk than the free cash flows.

We argue that it is more realistic to assume that a company maintains a fixed book-value leverage ratio than to assume, as Miles-Ezzell (1980) do, that the company maintains a fixed market-value leverage ratio.

Table I
Value today of the increases of debt implicit in the most popular formulae
for calculating the value of tax shields.
Perpetuities growing at a constant rate g

Authors	VTS_0	$PV_0[\Delta D_t] = \sum_1^\infty E[M_t \Delta D_t]$
Miles-Ezzell (1980) Arzac-Glosten (2005)	$\dfrac{D_0 R_F T}{(Ku - g)} \dfrac{(1 + Ku)}{(1 + R_F)}$	$\dfrac{D_0}{Ku - g}\left(g - \dfrac{Ku - R_F}{1 + R_F}\right)$
Modigliani-Miller (1963)	$\dfrac{D_0 R_F T}{(R_F - g)}$	$\dfrac{g \cdot D_0}{R_F - g}$
Fernández (2004)	$\dfrac{D_0 Ku T}{(Ku - g)}$	$\dfrac{g \cdot D_0}{Ku - g}$
Constant book-value leverage	$\dfrac{D_0 \alpha T}{(\alpha - g)}$	$\dfrac{g \cdot D_0}{\alpha - g}$

Ku = unlevered cost of equity
T = corporate tax rate
D_0 = debt value today
R_F = risk-free rate
α = required return to the increases of assets

Table II
Main formulas in the appendixes for growing perpetuities

	Modigliani-Miller D_t fixed $\Delta D_t = K \cdot CFd_t$	Fernández (2004) $D_t = K \cdot Ebv_t$ $\Delta D_t = K \cdot FCF_t$	Miles-Ezzell $D_t = K \cdot E_t$ $\Delta D_t = K \cdot \Delta FCF_t$
VTS_0	$\dfrac{D_0 R_F T}{(R_F - g)}$	$\dfrac{D_0 Ku T}{(Ku - g)}$	$\dfrac{D_0 R_F T}{(Ku - g)} \dfrac{(1 + Ku)}{(1 + R_F)}$
$1 + K_{\Delta D1}$	$1 + R_F$	$1 + Ku$	$\dfrac{g(1 + Ku)(1 + R_F)}{g(1 + R_F) + R_F - Ku}$
$K_{\Delta D2}$	R_F	Ku	Ku
Ke	$Ku + \dfrac{D_0}{E_0}(Ku - R_F)(1 - \dfrac{R_F T}{(R_F - g)})$	$Ku + \dfrac{D_0}{E_0}(Ku - R_F)(1 - T)$	$Ku + \dfrac{D_0}{E_0}(Ku - R_F)\left[1 - \dfrac{R_F T}{1 + R_F}\right]$
K_{TS1}	R_F	R_F	R_F
K_{TS2}	R_F	$\dfrac{R_F(1 + Ku) + gKu(1 + R_F)}{(1 + Ku) + g(1 + R_F)}$	Ku

	$1 + Ke_1$	
Miles-Ezzell	$\dfrac{(Ke - g)(1 + Ku)}{(Ku - g)}$	(A1.6)
Modigliani-Miller	$\dfrac{(Ke - g)(1 + R_F)(1 + Ku)}{(Ku - g)(1 + R_F) + (Ke - Ku)(1 + g)}$	(A3.3)
Fernández (2004)	$\dfrac{(Ke - g)(1 + R_F)(1 + Ku)}{(Ku - g)(1 + R_F) + (Ke - Ku)}$	(A2.6)
$D_t = L \cdot Ebv_t$	$\dfrac{E_0(Ke - g)}{Vu_0 \dfrac{(Ku - g)}{(1 + Ku)} + D_0\left[\dfrac{g}{(1 + \alpha)} - \dfrac{R_F(1 - T)}{(1 + R_F)}\right]}$	(A2.5)

Table III
Example. Valuation of a constant growing company
$FCF_0 = 60$; $A_0 = 1,000$; $D_0 = 500$;

$R_F = 4\%$; $Ku = 9\% = \alpha$; $T = 40\%$; $g = 3\%$; $Vu_0 = 1,030$

	Modigliani-Miller	Fernández	Miles-Ezzell
	D_t fixed	$D_t = K \cdot Ebv_t$	$D_t = K \cdot E_t$
	$\Delta D_t = K \cdot CFd_t$	$\Delta D_t = K \cdot FCF_t$	$\Delta D_t = K \cdot \Delta FCF_t$
VTS_0	800	300	139.74
E_0	1,330	830	669.74
$PV_0[\Delta D_t]$	1,500	250	-150.64
$Gu = PV_0[TAX_{Ut}]$	1,020	1,020	1,020
$G_L = PV_0[TAX_{Lt}]$	220	720	880.26
Ke average	7.87%	10.81%	12.68%

	Modigliani-Miller D_t fixed		Fernández $D_t = K \cdot Ebv_t$		Miles-Ezzell $D_t = K \cdot E_t$	
	t=1	t=2	t=1	t=2	t=1	t=2
$K_{\Delta Dt}$	4%	4%	9%	9%	-280.9%	9%
K_{Dt}	4%	4%	4.14%	4.27%	9%	9%
Ke_t	8.76%	8.75%	9.98%	10.01%	75.77%	9%
K_{St}	7.83%	7.79%	10.87%	10.94%	9%	9%
K_{Vut}	9%	9%	9%	9%	9%	9%
K_{VTSt}	4%	4%	5.711%	5.753%	9%	9%
K_{TSt}	4%	4%	4%	4.139%	4%	9%
K_{TAXUt}	9%	9%	9%	9%	9%	9%
K_{TAXLt}	9.79%	9.84%	9.79%	9.81%	9.79%	9%

Table IV
Value of the tax shields (VTS) according to the different theories as a function of g (expected growth) and α (required return to the increase of assets).
$D_0 = 500$; $R_F = 4\%$; $Ku = 9\%$; $T = 40\%$

	g					
	0%	1%	2%	3%	4%	5%
Miles-Ezzell	93.16	104.81	119.78	139.74	167.69	209.62
Modigliani-Miller	200.00	266.67	400.00	800.00	∞	∞
Fernández (2004)	200.00	225.00	257.14	300.00	360.00	450.00
$D_t = L \cdot Ebv_t$; α=5%	200.00	250.00	333.33	500.00	999.93	9476.19
$D_t = L \cdot Ebv_t$; α=7%	200.00	233.33	280.00	350.00	466.67	700.00
$D_t = L \cdot Ebv_t$; α=9%	200.00	225.00	257.14	300.00	360.00	450.00
$D_t = L \cdot Ebv_t$; α=11%	200.00	220.00	244.44	275.00	314.29	366.67
$D_t = L \cdot Ebv_t$; α=15%	200.00	214.29	230.77	250.00	272.73	300.00

Table V
Value today of the increases of debt in different periods and the sum of all of them
$D_0 = 500$; $R_F = 4\%$; $Ku = 9\%$; $T = 40\%$; $g = 3\%$

$PV_0(\Delta D_t)$	t=1	t=2	t=3	t=4	t=5	t=10	t=20	t=30	t=40	t=50	Sum
Miles-Ezzell	-8.29	-7.84	-7.40	-7.00	-6.61	-4.98	-2.83	-1.61	-0.91	-0.52	-150.64
Modigliani-Miller	14.42	14.28	14.15	14.01	13.88	13.22	12.00	10.90	9.89	8.98	1,500.00
Fernández (2004)	13.76	13.00	12.29	11.61	10.97	8.27	4.69	2.66	1.51	0.86	250.00
$D_t = L \cdot Ebv_t$; α=5%	14.29	14.01	13.75	13.48	13.23	12.02	9.91	8.18	6.75	5.57	750.00
$D_t = L \cdot Ebv_t$; α=7%	14.02	13.49	12.99	12.50	12.04	9.95	6.80	4.64	3.17	2.17	375.00
$D_t = L \cdot Ebv_t$; α=9%	13.76	13.00	12.29	11.61	10.97	8.27	4.69	2.66	1.51	0.86	250.00
$D_t = L \cdot Ebv_t$; α=11%	13.51	12.54	11.64	10.80	10.02	6.89	3.26	1.54	0.73	0.35	187.50

Table VI
Appropriate discount rates for the expected values of the equity cash flows (Ke_t)
$FCF_0 = 60$; $D_0 = 500$; $R_F = 4\%$; $Ku = 9\%$; $T = 40\%$; $g = 3\%$

Ke_t	t=1	t=2	t=3	t=5	t=10	t=20	t=30	t=50
Miles-Ezzell	75.77%	9%	9%	9%	9%	9%	9%	9%
Modigliani-Miller	8.76%	8.75%	8.74%	8.72%	8.71%	8.64%	8.45%	8.17%
Fernández (2004)	9.98%	10.01%	10.03%	10.06%	10.09%	10.26%	10.71%	11.46%
$D_t = L \cdot Ebv_t$; α=5%	9.01%	9.01%	9.02%	9.02%	9.03%	9.05%	9.13%	9.26%
$D_t = L \cdot Ebv_t$; α=7%	9.50%	9.52%	9.55%	9.57%	9.60%	9.74%	10.13%	10.78%
$D_t = L \cdot Ebv_t$; α=9%	9.98%	10.01%	10.03%	10.06%	10.09%	10.26%	10.71%	11.46%
$D_t = L \cdot Ebv_t$; α=11%	10.44%	10.46%	10.48%	10.50%	10.52%	10.63%	10.98%	11.57%

Table VII
Appropriate discount rates for the expected value of the taxes of the levered company.
$\alpha = Ku = 9\%$; $FCF_0 = 60$; $D_0 = 500$; $R_F = 4\%$; $T = 40\%$; $g = 3\%$

K_{TAXLt}	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8
Miles-Ezzell	9.79%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
Modigliani-Miller	9.79%	9.84%	9.89%	9.94%	9.99%	10.05%	10.11%	10.18%
Fernández (2004)	9.79%	9.81%	9.83%	9.86%	9.88%	9.90%	9.92%	9.95%

Table VIII
Ku as a function of g (growth) and α (required return to the increase of assets) if the required return to the profit after tax of the unlevered company (K_{PATu}) is fixed
$K_{PATu} = 9\%$; $FCF_0 = 60$; $D_0 = 500$; $R_F = 5\%$; $T = 40\%$

		g					
		0%	1%	2%	3%	4%	5%
α	7%	9.00%	9.47%	10.05%	10.92%	12.73%	24.38%
	8%	9.00%	9.19%	9.40%	9.65%	9.95%	10.44%
	9%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
	10%	9.00%	8.86%	8.73%	8.61%	8.52%	8.45%
	12%	9.00%	8.66%	8.38%	8.16%	8.03%	7.98%
	15%	9.00%	8.47%	8.08%	7.83%	7.70%	7.71%

Appendix 1

Derivation of formulas for Miles-Ezzell: $D_t = L\,S_t$

We are valuing a company with no leverage cost. The cost of debt is the risk-free rate (R_F). The company is a growing perpetuity, which means that $E_0[D_t] = D_0\,(1+g)^t$. If $D_t = L\,S_t$:

$$S_t + D_t = V_{Lt} = (1+L)S_t = \frac{1+L}{L}D_t \qquad (A1.1)$$

A solution for this valuation must be:

$$D_t + S_t = b \cdot FCF_t \qquad (A1.2)$$

A1.1. Derivation of $\sum_1^\infty E[M_t \Delta D_t]$

According to (A1.1) and (A1.2),

$$D_t = [L/(1+L)]\,V_{Lt} = [L/(1+L)]\,b\,FCF_t \qquad (A1.3)$$

Then, using (11):

$$E_0[M_{0,1}\cdot \Delta D_1] = E_0[M_{0,1}\cdot(D_1 - D_0)] = E_0\left[M_{0,1}\cdot\frac{L}{1+L}bFCF_1\right] - \frac{D_0}{1+R_F} = \frac{L\,b}{1+L}\frac{FCF_0(1+g)}{1+Ku} - \frac{D_0}{1+R_F}$$

As, according to (A1.3), $b\,FCF_0 = [(1+L)/L]\,D_0$:

$$E_0[M_{0,1}\cdot\Delta D_1] = \frac{D_0(1+g)}{1+Ku} - \frac{D_0}{1+R_F} \qquad (32)$$

For $t > 1$:

$$E_0[M_{0,t}\cdot\Delta D_t] = E_0[M_{0,t}\cdot(D_t - D_{t-1})] = E_0\left[M_{0,t}\cdot\frac{L}{1+L}b(FCF_t - FCF_{t-1})\right] =$$

$$= \frac{L}{1+L}b\,FCF_0\left(\frac{(1+g)^t}{(1+Ku)^t} - \frac{(1+g)^{t-1}}{(1+Ku)^{t-1}(1+R_F)}\right)$$

As $b\,FCF_0 = [(1+L)/L]\,D_0$:

$$E_0[M_{0,t}\cdot\Delta D_t] = D_0\frac{(1+g)^{t-1}}{(1+Ku)^{t-1}}\left(\frac{(1+g)}{(1+Ku)} - \frac{1}{(1+R_F)}\right) \qquad (33)$$

$$\sum_1^\infty E[M_t\Delta D_t] = D_0\left(\frac{1+g}{1+Ku} - \frac{1}{1+R_F}\right) + ... + D_0\frac{(1+g)^{t-1}}{(1+Ku)^{t-1}}\left(\frac{(1+g)}{(1+Ku)} - \frac{1}{(1+R_F)}\right) + ... \qquad (A1.4)$$

(A1.4) is the sum of a geometric progression with growth rate $= (1+g)/(1+Ku)$. Then:

$$\sum_1^\infty E[M_t\Delta D_t] = D_0\left(\frac{1+g}{1+Ku} - \frac{1}{1+R_F}\right)\frac{1+Ku}{Ku-g} = \frac{D_0}{Ku-g}\left(1+g - \frac{1+Ku}{1+R_F}\right) = \frac{D_0}{Ku-g}\left(g - \frac{Ku-R_F}{1+R_F}\right) \qquad (34)$$

Appendix 1 (continued)

A1.2. Derivation of VTS

Substituting (34) in (3), we get:

$$VTS_t = \frac{D_t R_F T}{(Ku - g)} \frac{(1 + Ku)}{(1 + R_F)} \qquad (35)$$

A.1.3. Relationship among the increases of debt and the free cash flows

As, according to equation (4), $V_{L0} = Vu_0 + VTS_0$, using equations (A1.3), (14) and

(35): $V_{L0} = bFCF_0 = \frac{D_0 R_F T}{(Ku - g)} \frac{(1 + Ku)}{(1 + R_F)} + \frac{FCF_0(1 + g)}{(Ku - g)} = \frac{L}{1 + L} b \frac{FCF_0 R_F T}{(Ku - g)} \frac{(1 + Ku)}{(1 + R_F)} + \frac{FCF_0(1 + g)}{(Ku - g)}$

Solving for b, we find:

$$b = \frac{(1 + g)}{(Ku - g) - \frac{L}{(1 + L)} R_F T \frac{1 + Ku}{1 + R_F}} \qquad (A1.5)$$

After equation (A1.4), it is obvious that:

$$\Delta D_t = [L/(1+L)] \, b \, \Delta FCF_t \qquad (A1.6)$$

$$\Delta D_t = \frac{(1 + g)}{\frac{(Ku - g)(1 + L)}{L} - R_F T \frac{1 + Ku}{1 + R_F}} \Delta FCF_t = \frac{D_0}{FCF_0} \Delta FCF_t \qquad (A1.7)$$

It is clear that under Miles-Ezzell we assume that the increases of debt are proportional to the increases of the free cash flows.

A.1.4. Equivalent discount rate for the increases of debt in different periods ($K_{\Delta D}$)

The equivalent discount rate for the expected increase of debt in period 1 ($K_{\Delta D1}$) implied by (32) is:

$$E_0[M_{0,1} \cdot \Delta D_1] = \frac{D_0(1 + g)}{1 + Ku} - \frac{D_0}{1 + R_F} = \frac{E_0[\Delta D_1]}{1 + K_{\Delta D1}} = \frac{gD_0}{1 + K_{\Delta D1}} \qquad (A1.8)$$

Some algebra permits to express $1 + K_{\Delta D1} = \frac{g(1 + Ku)(1 + R_F)}{g(1 + R_F) + R_F - Ku}$ (A1.9)[8]

Equation (A1.9) is asymptotic in $g = (Ku - R_F)/(1 + R_F)$. In this situation, we know from equation (32) that $E_0[M_{0,1} \cdot \Delta D_1] = 0$.

$$E_0[M_{0,2} \cdot \Delta D_2] = D_0 \frac{(1 + g)}{(1 + Ku)} \left(\frac{(1 + g)}{(1 + Ku)} - \frac{1}{(1 + R_F)} \right) = \frac{g D_0 (1 + g)}{(1 + K_{\Delta D1})(1 + K_{\Delta D2})}$$

---

[8] Note that if g=0, then $K_{\Delta D1}=-100\%$. This does not make any economic sense because in this situation the expected value of the increase of debt is also 0 and $E_0[M_{0,1} \cdot \Delta D_1] = \frac{D_0}{1 + Ku} - \frac{D_0}{1 + R_F}$.

Appendix 1 (continued)

After equation (A1.8) it is obvious that $K_{\Delta D2} = Ku$. Repeating this exercise, we find that $K_{\Delta Dt} = Ku$. The appropriate discount rates for the expected increases of debt are different for $t = 1$ and for the following periods.

A.1.5. Value today of the expected taxes

The present values at t=0 of equations (38) and (39) are:

$$Gu_0 = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot FCF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t]\right) = \left(\frac{T}{1-T}\right)\left(Vu_0 + \sum_1^\infty E[M_t \cdot \Delta A_t]\right) \qquad (40)$$

$$G_{L0} = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot ECF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t] - \sum_1^\infty E[M_t \cdot \Delta D_t]\right) \qquad (41)$$

A.1.6. The value of tax shields is the difference between Gu and G_L

We want to prove that $VTS_0 = Gu_0 - G_{L0}$. Subtracting (41) from (40), we get:

$$VTS_0 = Gu_0 - G_{L0} = \left(\frac{T}{1-T}\right)\left(Vu_0 - S_0 + \sum_1^\infty E[M_t \cdot \Delta D_t]\right) \qquad (A1.10)$$

As, according to (4), $Vu_0 - S_0 = D_0 - VTS_0$

$$VTS_0 = \left(\frac{T}{1-T}\right)\left(D_0 - VTS_0 + \sum_1^\infty E[M_t \cdot \Delta D_t]\right) \qquad (A1.11)$$

Equation (A1.11) is identical to equation (3)

A.1.7. Appropriate discount rate for the expected taxes

The appropriate discount rate for the expected taxes of the unlevered company is:

$$PV_0[Taxes_{U1}] = \frac{E_0[Taxes_{U1}]}{(1+K_{TAXU1})} = \left[\frac{T}{1-T}\right]\left(\frac{(1+g)FCF_0}{(1+Ku)} + \frac{gA_0}{(1+\alpha)}\right)$$

As $E[Taxes_{U1}] = [T/(1-T)] [FCF_0(1+g) + gA_0]$, we can calculate K_{TAXU1}.

$$(1+K_{TAXU1}) = \frac{E_0[Taxes_{U1}]}{PV_0[Taxes_{U1}]} = \frac{(1+g)FCF_0 + gA_0}{(1+g)FCF_0(1+\alpha) + gA_0(1+Ku)}(1+Ku)(1+\alpha) \qquad (A1.12)$$

If $\alpha = Ku$, then $K_{TAXUt} = Ku$

The appropriate discount rate for the expected taxes of the levered company is:

$$PV_0[Taxes_{L1}] = \frac{E_0[Taxes_{L1}]}{(1+K_{TAXL1})} = \left[\frac{T}{1-T}\right]\left(\frac{(1+g)FCF_0}{(1+Ku)} + \frac{gA_0}{(1+\alpha)} - \frac{D_0R_F(1-T)}{(1+R_F)}\right)$$

Appendix 1 (continued)

As $E[Taxes_{L1}] = [T/(1-T)][FCF_0(1+g) + gA_0 - D_0 R_F (1-T)]$

$$1 + K_{TAXL1} = \frac{(1+g)FCF_0 + gA_0 - D_0 R_F(1-T)}{\dfrac{(1+g)FCF_0}{(1+Ku)} + \dfrac{gA_0}{(1+\alpha)} - \dfrac{D_0 R_F(1-T)}{(1+R_F)}}$$ (A1.13)

For t > 1, (for example, for t=2), the present value is:

$$PV_0[Taxes_{L2}] = \frac{E_0[Taxes_{L1}](1+g)}{(1+K_{TAXL1})(1+K_{TAXL2})}$$

It is obvious that $K_{TAXL2} = Ku$ if $\alpha = Ku$

From equation (11) we can calculate the present value of the levered taxes:

$$G_{L0} = Gu_0 - VTS_0 = \frac{T}{1-T}\left[Vu_0 + \frac{g A_0}{(\alpha - g)}\right] - \frac{D_0 R_F T}{(Ku-g)}\frac{(1+Ku)}{(1+R_F)}$$ (A1.14)

Although K_{TAXUt} and K_{TAXLt} are not constant, we can calculate K_{TAXU} and K_{TAXL} such that $G_{U0} = Taxes_{U0} (1+g) / (K_{TAXU} - g)$ and $G_{L0} = Taxes_{L0} (1+g) / (K_{TAXL} - g)$. Some algebra permits to find:

$$K_{TAXU} = \frac{Vu_0(\alpha - g)Ku + g\alpha A_0}{Vu_0(\alpha - g) + gA_0}$$ (A1.15)

$$K_{TAXL} = g + \frac{S_0(Ke - g) + g(A_0 - D_0)}{Vu_0 + \dfrac{gA_0}{(\alpha - g)} - \dfrac{VTS_0(1-T)}{T}}$$ (A1.16)

A.1.8. Required return for the expected equity cash flow in period 1 (Ke$_1$)

The value in period t of the equity value in period t+1 is:

$$E_t[M_{t,t+1}\cdot S_{t+1}] = E_t[M_{t,t+1}\cdot b\cdot FCF_{t+1}/(1+L)] = \frac{b}{1+L}\frac{FCF_t(1+g)}{(1+Ku)} = \frac{S_t(1+g)}{(1+Ku)}$$ (A1.17)

The value of the equity value today is:

$$S_0 = E_0[M_{0,1}\cdot ECF_1] + E_0[M_{0,1}\cdot S_1] = E_0[M_{0,1}\cdot ECF_1] + \frac{S_0(1+g)}{1+Ku}$$ (A1.18)

The equivalent discount rate for the expected equity cash flow in period 1 (Ke1) is:

$$E_0[M_{0,1}\cdot ECF_1] = \frac{E_0[ECF_1]}{1+Ke_1} = \frac{ECF_0(1+g)}{1+Ke_1} = \frac{S_0(Ku-g)}{1+Ku}$$

$$1 + Ke_1 = \frac{ECF_0(1+g)(1+Ku)}{S_0(Ku-g)}$$ (A1.19)

Appendix 1 (continued)

A.1.9. Required return for the expected equity cash flow in period t>1 (Ke_t)

The value today of the equity cash flow received in period 2 is:

$$E_0[M_{0,2}{\cdot}ECF_2] = \frac{E_0[ECF_2]}{(1+Ke_1)(1+Ke_2)} = \frac{ECF_0(1+g)^2}{(1+Ke_1)(1+Ke_2)} = \frac{E_0[M_{0,1}{\cdot}ECF_1](1+g)}{(1+Ke_2)}$$

It is obvious that

$$1+Ke_2 = \frac{E_0[M_{0,1}{\cdot}ECF_1](1+g)}{E_0[M_{0,2}{\cdot}ECF_2]} \qquad (A1.20)$$

The value of the equity value today is:

$$S_0 = E_0[M_{0,1}{\cdot}ECF_1] + E_0[M_{0,2}{\cdot}ECF_2] + E_0[M_{0,2}{\cdot}S_2] = \frac{S_0(Ku-g)}{(1+Ku)} + E_0[M_{0,2}{\cdot}ECF_2] + \frac{S_0(1+g)^2}{(1+Ku)^2}$$

$$E_0[M_{0,2}{\cdot}ECF_2] = S_0(1 - \frac{(Ku-g)}{(1+Ku)} - \frac{(1+g)^2}{(1+Ku)^2}) = S_0(\frac{(1+g)}{(1+Ku)} - \frac{(1+g)^2}{(1+Ku)^2})$$

$$1+Ke_2 = \frac{S_0\dfrac{(Ku-g)}{(1+Ku)}(1+g)}{S_0(\dfrac{(1+g)}{(1+Ku)} - \dfrac{(1+g)^2}{(1+Ku)^2})} = \frac{(Ku-g)}{1 - \dfrac{(1+g)}{(1+Ku)}} = 1+Ku \qquad (A1.21)$$

Following the same procedure, it may be shown that for t >1, $Ke_t = Ku$.

A.1.10. Average required return for the expected equity cash flows (Ke)

We want to find an average required return for the expected equity cash flows (Ke) such that:

$$S_0 = \sum_1^\infty E[M_t{\cdot}ECF_t] = \frac{ECF_0(1+g)}{Ke-g} \qquad (A1.22)$$

The relationship between expected values at t=1 of the free cash flow, the equity cash flow and the debt cash flow (CFd) is:

$$S_0(Ke\text{-}g) = ECF_0(1+g) = Vu_0(Ku\text{-}g) - D_0(R_F - g) + D_0\,R_F\,T$$
As $E_0 = Vu_0 - D_0 + VTS_0$, then $S_0\,Ke = Vu_0\,Ku - D_0\,R_F + VTS_0\,g + D_0\,R_F\,T$

And the general equation for Ke is:

$$Ke = Ku + \frac{D_0}{S_0}[Ku - R_F(1-T)] - \frac{VTS_0(Ku-g)}{S_0} \qquad (A1.23)$$

Substituting (35) in (A1.23):

$$Ke = Ku + \frac{D_0}{S_0}(Ku - R_F)\left[1 - \frac{R_F T}{1+R_F}\right] \qquad (A1.24)$$

But this expression is the average Ke. It is not the required return to equity (Ke_t) for all periods.

Appendix 1 (continued)

A.1.11. Formulas with continuous adjustment of debt

If debt is adjusted continuously, not only at the end of the period, then the formula (35) changes to

$$VTS_0 = \int_0^\infty T\rho D_t e^{(\gamma-\kappa)t} dt = \frac{D_0\,\rho\,T}{\kappa-\gamma} \qquad (A1.25)$$

where $\rho = \ln(1+R_F)$, $\gamma = \ln(1+g)$, and $\kappa = \ln(1+Ku)$.

Perhaps formula (A1.25) induces Cooper and Nyborg (2006) and Ruback (1995 and 2002) to use (A1.26) as the expression for the value of tax shields when the company maintains a constant market-value leverage ratio ($D_t = L\,S_t$):

$$VTS_0 = \frac{D_0\,R_F\,T}{Ku-g} \qquad (A1.26)$$

But (A1.26) is incorrect for discrete time: (35) is the correct formula.

If $D_t = L{\cdot}S_t$, the appropriate discount rate for the expected value of the unlevered equity (Vu_t), for the expected value of the debt (D_t), for the expected value of the tax shields (VTS_t), and for the expected value of the equity (S_t) is Ku in all periods.

$D_t = L{\cdot}E_t$ is absolutely equivalent to $D_t = M{\cdot}Vu_t$. In both cases $\Delta D_t = X{\cdot}\Delta FCF_t$, where $X = D_0 / FCF_0$.

Appendix 2

Derivation of formulas if debt is proportional to the book value of equity

$D_t = K \cdot Ebv_t$, where Ebv is the book value of equity. Then $\Delta D_t = K \cdot \Delta Ebv_t$ and the relationship between ΔD_t and ΔA_t (increase of assets) is[9] $\Delta D_t = \Delta A_t / (1+1/K)$.

A2.1. $\sum_1^\infty E[M_t \Delta D_t]$

In section 2.1 we derive equation (22):

$$E_0[M_{0,t} \cdot \Delta D_t] = \Delta D_0 \frac{(1+g)^t}{(1+\alpha)^t} \tag{22}$$

A2.2. VTS

In section 2.1 we derive equation (24):

$$VTS_0 = \frac{D_0 \alpha T}{(\alpha - g)} \tag{24}$$

A.2.3. Relationship among the increases of debt and the free cash flows

According to equation (18), the increases of debt are proportional to the increases of assets:

$$\Delta D_t = \Delta A_t / (1+1/K) \tag{18}$$

A.2.4. Equivalent discount rate for the increases of debt in different periods ($K_{\Delta D}$)

According to equation (22), the equivalent discount rate for the increases of debt is α for all periods.

A.2.5. Value today of the expected taxes

Equations (40) and (41), in this case, are:

$$Gu_0 = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot FCF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t]\right) = \left(\frac{T}{1-T}\right)\left(Vu_0 + \frac{gA_0}{\alpha - g}\right) \tag{40}$$

$$G_{L0} = \left(\frac{T}{1-T}\right)\left(\sum_1^\infty E[M_t \cdot ECF_t] + \sum_1^\infty E[M_t \cdot \Delta A_t] - \sum_1^\infty E[M_t \cdot \Delta D_t]\right) = \left(\frac{T}{1-T}\right)\left(S_0 + \frac{g(A_0 - D_0)}{\alpha - g}\right) \tag{41}$$

A.2.6. The value of tax shields is the difference between Gu and G_L

Subtracting (41) from (40), we get the same result as in (24):

$$VTS_0 = Gu_0 - G_{L0} = \left(\frac{T}{1-T}\right)\left(Vu_0 - S_0 + \frac{gD_0}{\alpha - g}\right) = \left(\frac{T}{1-T}\right)\left(D_0 - VTS_0 + \frac{gD_0}{\alpha - g}\right)$$

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[9] The increase of the book value of equity is equal to the profit after tax (PAT) minus the equity cash flow. According to equation (15): $\Delta Ebv_t = PAT_{Lt} - ECF_t = \Delta A_t - \Delta D_t = \Delta D_t / K$.
In this situation, the increase of debt is proportional to the increases of net assets, and the risk of the increases of debt is equal to the risk of the increases of assets: $\Delta D_t = \Delta A_t / (1+1/K)$

Appendix 2 (continued)

A.2.7. Appropriate discount rate for the expected taxes

(A1.12), (A1.13) and (A1.14) also apply to this situation. K_{TAXU} is defined by equation (A1.15). If $\alpha = Ku$, $K_{TAXU} = Ku$, and K_{TAXL}:

$$K_{TAXL} = Ku + \frac{D_0(1-T)(Ku-R_F)(Ku-g) + g(A_0-D_0)}{Vu_0(Ku-g) + gA_0 - D_0Ku(1-T)}$$ (A2.1)

A.2.8. Required return for the expected equity cash flow in period 1 (Ke_1)

Calculating the expected value in t=0 for a growing perpetuity and substituting the value of $K_{\Delta D} = \alpha$:

$$E_0[M_{0,1} \cdot ECF_1] = \frac{(1+g)ECF_0}{(1+Ke_1)} = \frac{(1+g)FCF_0}{(1+Ku)} - \frac{D_0R_F(1-T)}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$\frac{S_0(Ke-g)}{(1+Ke_1)} = \frac{Vu_0(Ku-g)}{(1+Ku)} - \frac{D_0R_F(1-T)}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$(1+Ke_1) = \frac{S_0(Ke-g)}{Vu_0\dfrac{(Ku-g)}{(1+Ku)} + D_0\left[\dfrac{g}{(1+\alpha)} - \dfrac{R_F(1-T)}{(1+R_F)}\right]}$$ (A2.2)

If $\alpha = Ku$: $(1+Ke_1) = \dfrac{(Ke-g)(1+R_F)(1+Ku)}{(Ku-g)(1+R_F) + (Ke-Ku)}$ A2.3)

A.2.9. Required return for the expected equity cash flow in period t>1 (Ke_t)

(A1.20) is also valid in this situation

$$1+Ke_2 = \frac{E_0[M_{0,1} \cdot ECF_1](1+g)}{E_0[M_{0,2} \cdot ECF_2]}$$ (A1.20)

A.2.10. Average required return for the expected equity cash flows (Ke)

Substituting (23) in (A1.23):

$$Ke = Ku + \frac{D_0}{E_0}\left[(Ku-R_F(1-T)) - \frac{\alpha T(Ku-g)}{(\alpha-g)}\right]$$ (A2.4)

If $\alpha = Ku$: $Ke = Ku + \dfrac{D_0}{E_0}(1-T)(Ku-R_F)$

This expression is the average Ke: It is not the required return to equity (Ke_t) for all periods.

A.2.11. The appropriate discount rate for the expected value of the tax shields

The tax shield of the next period is known with certainty ($D_0 R_F T$) and the appropriate discount rate is R_F. The appropriate discount rate for the expected tax shield of t = 2 (K_{TS2}) is:

Appendix 2 (continued)

$$E_0\left[M_{0,2}\cdot D_1 R_F T\right] = \frac{D_0(1+g)R_F T}{(1+R_F)(1+K_{TS2})} = R_F T\left[\frac{D_0}{(1+R_F)^2} + \frac{gD_0}{(1+R_F)(1+\alpha)}\right]$$

$$K_{TS2} = \frac{R_F(1+\alpha) + g\alpha(1+R_F)}{(1+\alpha) + g(1+R_F)} \qquad (A2.5)$$

The present value of the tax shield of period t is:

$$E_0\left[M_{0,t}\cdot D_{t-1} R_F T\right] = \frac{D_0(1+g)^{t-1} R_F T}{(1+R_F)(1+K_{TS2})...(1+K_{TSt})} = \frac{D_0 R_F T}{(1+R_F)^t} + \frac{gD_0 R_F T}{(1+R_F)^{t-1}(1+\alpha)} + ... + \frac{gD_0(1+g)^{t-2} R_F T}{(1+R_F)(1+\alpha)^{t-1}}$$

This expression is the sum of a geometric progression with a factor $X = (1+g)(1+R_F)/(1+\alpha)$. The solution is:

$$PV_0\left[TS_t\right] = \frac{R_F TD_0}{(1+R_F)^{t-1}}\left[\frac{1}{(1+R_F)} + \frac{g\left[X^{t-1}-1\right]}{(1+\alpha)\left[X-1\right]}\right] \qquad (A2.6)$$

$$(1+K_{TSt}) = \frac{PV_0\left[TS_{t-1}\right]}{PV_0\left[TS_t\right]}(1+g) = X(1+\alpha)\frac{(1+\alpha)(X-1) + g(1+R_F)(X^{t-2}-1)}{(1+\alpha)(X-1) + g(1+R_F)(X^{t-1}-1)}$$

When t tends to infinity, $K_{TSt} = MIN[\alpha, (1+R_F)(1+g)-1]$

A.2.12. The appropriate discount rate for the expected value of tax shields (VTS)

$$VTS_0 = E_0\left[M_{0,1}\cdot D_0 R_F T\right] + E_0\left[M_{0,1}\cdot VTS_1\right] = \frac{D_0 TR_F}{(1+R_F)} + \frac{VTS_0(1+g)}{(1+K_{VTS1})}$$

$$(1+K_{VTS1}) = \frac{\alpha(1+R_F)(1+g)}{(\alpha + gR_F)}$$

A.2.13. The appropriate discount rate for the expected value of equity (S)

$$S_0 = E_0\left[M_{0,1}\cdot ECF_1\right] + E_0\left[M_{0,1}\cdot S_1\right] = \frac{ECF_0(1+g)}{(1+Ke_1)} + \frac{S_0(1+g)}{(1+K_{S1})}$$

$$(1+K_{S1}) = \frac{S_0(1+g)}{S_0 - ECF_0(1+g)/(1+Ke_1)}$$

A.2.14. The appropriate discount rate for the expected value of debt (D)

$$E_0\left[M_{0,1}\cdot D_1\right] = \frac{(1+g)D_0}{(1+K_{D1})} = \frac{D_0}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$(1+K_{D1}) = \frac{(1+g)(1+R_F)(1+\alpha)}{1+\alpha + g(1+R_F)}$$

Appendix 3

Derivation of formulas if D_t is known with certainty at t=0.

This is Modigliani-Miller's assumption. This situation can be analyzed as one special case of Appendix 2: that of a company with $\alpha = R_F$.

The appropriate discount for the expected value of the equity cash flow

Substituting (30) in (A1.23), or substituting α by R_F in (A2.4):

$$\text{Ke} = \text{Ku} + \frac{D_0}{S_0}\left[(\text{Ku} - R_F(1-T) - R_F T\frac{(\text{Ku - g})}{(R_F \text{ - g})}\right] \tag{A3.1}$$

But this expression is the average Ke. It is not the required return to equity (Ke_t) for all the periods. Substituting α by R_F in (A2.2):

$$(1+\text{Ke}_1) = \frac{S_0(\text{Ke}-g)(1+\text{Ku})(1+R_F)}{\text{Vu}_0(\text{Ku}-g)(1+R_F)+D_0(1+\text{Ku})\left[g-R_F(1-T)\right]} \tag{A3.2}$$

It may be expressed also as: $(1+\text{Ke}_1) = \dfrac{(\text{Ke}-g)(1+\text{Ku})(1+R_F)}{(\text{Ku}-g)(1+R_F)+(\text{Ke}-\text{Ku})(1+g)}$ (A3.3)

For t=2, it is obvious that: $\text{PV}_0\left[\text{ECF}_2\right]= \text{PV}_0\left[\text{ECF}_1\right]\dfrac{(1+g)}{(1+\text{Ke}_2)}$

The appropriate discount rate for the expected taxes

(A1.12) and (A1.13) also apply to this situation.

For t = 2, the present value is: $\text{PV}_0\left[\text{Taxex}_{L2}\right]= \dfrac{E_0\left[\text{Taxex}_{L1}\right](1+g)}{(1+K_{\text{TAXL1}})(1+K_{\text{TAXL2}})}$

The appropriate discount rate for the expected value of tax shields (VTS)

$$\text{VTS}_0 = E_0\left[M_{0,1}{\cdot}D_0 R_F T\right]+E_0\left[M_{0,1}{\cdot}\text{VTS}_1\right]= \frac{D_0 R_F T}{(1+R_F)}+\frac{\text{VTS}_0\,(1+g)}{(1+K_{\text{VTS1}})}$$

It is obvious that $K_{\text{VTS1}} = R_F = K_{\text{VTSt}}$

The appropriate discount rate for the expected value of equity (S)

Calculating the present value of equation (1) at t =1:

$$E_0\left[M_{0,1}{\cdot}\text{VTS}_1\right]= E_0\left[M_{0,1}{\cdot}S_1\right]+ E_0\left[M_{0,1}{\cdot}D_1\right]- E_0\left[M_{0,1}{\cdot}\text{Vu}_1\right]$$

$$\text{VTS}_0\frac{(1+g)}{(1+R_F)}=S_0\frac{(1+g)}{(1+K_{S1})}+D_0\frac{(1+g)}{(1+R_F)}-\text{Vu}_0\frac{(1+g)}{(1+\text{Ku})}$$

$$\frac{S_0}{(1+K_{S1})}=\frac{\text{VTS}_0-D_0}{(1+R_F)}+\frac{\text{Vu}_0}{(1+\text{Ku})}=\frac{(\text{VTS}_0-D_0)(1+\text{Ku})+\text{Vu}_0(1+R_F)}{(1+R_F)(1+\text{Ku})}=\frac{S_0+(\text{VTS}_0-D_0)\text{Ku}+\text{Vu}_0 R_F}{(1+R_F)(1+\text{Ku})}$$

$$(1+K_{S1}) = \frac{S_0(1+R_F)(1+\text{Ku})}{(E_0-\text{Vu}_0)(1+\text{Ku})+\text{Vu}_0(1+R_F)}$$

Appendix 3 (continued)

For t=2

$$E_0[M_{0,2} \cdot S_2] = E_0[M_{0,2} \cdot (VTS_2 - D_2)] + E_0[M_{0,2} \cdot Vu_2]$$

$$\frac{S_0}{(1+K_{S1})(1+K_{S2})} = \frac{S_0 - Vu_0}{(1+R_F)^2} + \frac{Vu_0}{(1+Ku)^2} = \frac{(S_0 - Vu_0)(1+Ku)^2 + Vu_0(1+R_F)^2}{(1+R_F)^2(1+Ku)^2}$$

$$(1+K_{S1})(1+K_{S2}) = \frac{S_0(1+R_F)^2(1+Ku)^2}{(S_0 - Vu_0)(1+Ku)^2 + Vu_0(1+R_F)^2}$$

$$(1+K_{St}) = (1+R_F)(1+Ku)\frac{(S_0 - Vu_0)(1+Ku)^{t-1} + Vu_0(1+R_F)^{t-1}}{(S_0 - Vu_0)(1+Ku)^t + Vu_0(1+R_F)^t}$$

<u>The appropriate discount rate for the expected value of debt (D)</u>

$$E_0[M_{0,1} \cdot S_1] = \frac{(1+g)D_0}{(1+K_{D1})} = \frac{D_0}{(1+R_F)} + \frac{gD_0}{(1+R_F)} = \frac{(1+g)D_0}{(1+R_F)}$$

Then, $K_{Dt} = R_F$

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