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COMPETING FOR SHELF SPACE

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This paper studies competition for shelf space in a multi-supplier retail point. We consider a retailer that seeks to allocate her shelf space to maximize her profit. Because products associated with larger profit margin are granted more shelf space, suppliers can offer the retailer financial incentives to obtain larger space allocations. We analyze the competitive dynamics arising from the scarcity of space, and show existence and uniqueness of equilibrium. We then demonstrate that the inefficiencies from decentralizing decision-making are limited to 6% with wholesale-price contracts, and that full coordination can be achieved with pay-to-stay fee contracts. We finally investigate how competition is distorted under the practice of category management.

Key words: Game theory, Supply chain competition, Price of Anarchy, Pricing, Supply contracts.
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1. Introduction

Product proliferation has not only increased the complexity of manufacturing and distribution, it has also created new incentive issues in shelf space allocation. To cope with fiercer competition, manufacturers provide retailers with financial incentives to secure shelf space for their products. Consumer good manufacturers have indeed been reported to spend 15% of their revenue to pay stores to stock their products, totaling $100 billion per year in the United States (Forster 2002). The practice of these transfer payments remains nevertheless obscure, and its competitive nature and value to the end-consumer are highly debated. For instance, Hewlett-Packard Co. has recently been criticized for offering chain stores incentives to stop selling store-brand inkjet printer cartridges, in order to increase HP's cartridge's market share by reducing consumer choice (Hamm 2007). The Federal Trade Commission is in fact investigating whether paying for shelf space is anticompetitive, after small companies complained they were shut out of stores (FTC Report 2001).

In this paper, we analyze the shelf space allocation problem to understand how much pressure suppliers face to obtain shelf space. We define shelf space in a generic sense, including, among others, shelf space in a grocery store, parking spots at a car dealer, screens in a movie theater, and advertising space on a website. We assume that suppliers can obtain more shelf space from the retailer by conceding her larger profit margins, i.e., by lowering their wholesale prices. We model the capacity allocation as a game, demonstrate existence and uniqueness of a Nash equilibrium, and study its sensitivity to parameter changes. We also show that the inefficiencies created by the allocation mechanism are no greater than 6%, if the retail pricing decisions are exogenous, but can be up to 27% (specifically, $1-2/e$) otherwise. Finally, we discuss how the competitive outcome is affected under the following retailing practices: pay-to-stay fees, manufacturer's offering of an entire product category, introduction of store-brand products, and category management. Our simple model sheds light onto the potential benefits and pitfalls of various competitive strategies.

We assume that the number of products in the assortment is given, and we model the space allocated to each Stock Keeping Unit (SKU) as a continuous variable. For a review of assortment and shelf space models, see Kök et al. (2006). A key assumption of our model, first postulated by Lee (1961) and experimentally validated by Curhan (1973) and Dreze at al. (1994), is that, as shelf space is increased, unit sales increase at a decreasing rate. Under this assumption, Anderson (1979) and Corstjens and Doyle (1981, 1983) proposed a model for optimizing shelf space allocation across product categories, and solved it with geometric programming. Bultez and Naert (1988) developed a method for optimizing shelf space allocation among products within the same category, using an attraction model, and reported a 12% profit increase in a Belgian grocery store. If the capacity units are discrete, the shelf space allocation problem can be modeled as an integer optimization problem (Armstrong et al. 1982). Because these geometric and integer problems are complex to solve, especially with assortment decisions (Borin at al. 1995), meta-heuristics have been tailored to the shelf space allocation problem (e.g., Yang 2001 and Lim et al. 2004). Our demand model is a simplified version of the one proposed by Corstjens and Doyle (1981), to keep the analysis tractable, but maintains the same level of practicality: all parameters can easily be estimated with experimentation (e.g., Bultez and Naert 1988, Dreze et al. 1994) or cross-sectional methods of data collection (Corstjens and Doyle 1983, Van Dijk et al. 2004).

Building on this demand model, we focus on the pricing game among suppliers, taken the customer prices as fixed. With a similar model, Martín-Herrán et al. (2006) characterized the wholesale prices and shelf spaces in equilibrium with two competing suppliers. We complement their results by analyzing a case with an arbitrary number of suppliers and quantify the inefficiencies in the retail chain. Moreover, we provide proofs of their numerical observations and discuss the implications of alternative retail practices.

Among these widespread retail practices, we first consider pay-to-stay fees, which are a "rent" charged by retailers to suppliers in exchange for retailing space. We show that this mechanism improves supply chain efficiency, while reducing the suppliers' profits, refining a proposal made by Cairns (1962). Another notable type of slotting fee, which we do not consider in this paper, is the slotting allowance. Slotting allowances are lump-sum, up-front payment from a manufacturer to a retailer to have a new SKU carried on the retailer's shelves. Lariviere and Padmanabhan (1997) interpreted the slotting allowances as a signaling instrument about the potential sales of a new product. In contrast, pay-to-stay fees are charged for existing products to ensure continued presence on the shelf, and are therefore used more to cope with increased competition (through product proliferation, see Sullivan 1997) than to reduce demand uncertainty through signaling.

We then analyze supply chain integration, with a particular emphasis on category management. Under category management, the retailer delegates the category space allocation decision to one of the suppliers, typically the main player in category. The practice of category management is at the very least controversial and raises antitrust concerns (e.g., Steiner 2001, Bush and Gelb 2005). Zenor (1994) and Kurtuluş and Toktay (2005) compared the performance of a channel with and without category management, when demand is sensitive to prices but not to shelf space. In contrast, we assume that demand is sensitive to shelf space, consistently with Lee's observation, and consider the retail prices as exogenous. Zenor concluded from a case study and simulations that the benefits of category management can be substantial (as high as 30%) and are larger with more competition. Kurtuluş and Toktay analytically showed that category management improves customers' satisfaction, increases the retailer's profit, leaves the category captain indifferent, and decreases the other manufacturers' profit. Basuroy et al. (2001) derived similar results in a multi-brand, multi-retailer Cournot competitive model. Our model analysis corroborates their results, by showing that they hold even in the absence of pricing decisions.

Our work also relates to the abundant literature on supply chain coordination through supply contracts. With simple wholesale-price contracts, inventory decisions are typically not coordinated across the supply chain, a manifestation of the double-marginalization phenomenon (Spengler 1950). The limited performance of these contracts in the presence of stochastic demand was first investigated in two-stage supply chains by Lariviere and Porteus (2001) and Cachon and Lariviere (2001), then in more complex supply chains (see Cachon 2003 for a review). To improve coordination in supply chains, various alternative contracts have been proposed: buyback, revenue sharing, quantity flexibility, sales rebate, and quantity discount contracts (see the reviews by Cachon 2003 and Lariviere 1999). Compared to the large body of research on vertical interactions (through supply contracts) among supply chain partners, horizontal competition has received only limited attention. Under stochastic demand, horizontal competition has been studied in single-product distribution networks (e.g., Wang and Gerchak 2001 and Cachon 2003), multiple-product distribution networks (e.g., Bernstein and Federgruen 2005). Carr and Karmarkar (2005) and Kök (2006) analyzed assembly networks with a deterministic, price-quantity linear relationship and no capacity constraints, respectively focusing on supply network design and coordination mechanisms with supply contracts. In contrast to these papers, we explicitly model capacity constraints with multiple products, and show that inefficiencies arise even when retail prices are exogenous (and demand is deterministic).

To quantify supply chain efficiency, we use the *Price of Anarchy* (*PoA*), which measures the worst-case ratio of the profit of the integrated supply chain to the profit of the decentralized supply chain. The concept of *PoA* was introduced by Koutsoupias and Papadimitriou (1999), and has since then been extensively used in transportation networks, network resource allocation games, network pricing games, and supply chain games (see Martínez-de-Albéniz and Simchi-Levi 2003, Perakis and Roels 2007, and the references therein).

The remainder of the paper is organized as follows. In §2, we describe the model. We analyze the suppliers' game in §3. In particular, after showing existence and uniqueness of equilibrium, we quantify the inefficiencies created in the space allocation process and explore alternative modeling assumptions. We then discuss the impact of pay-to-stay fees and supply chain integration in §4 and conclude in §5. All the proofs are contained in the Appendix.

2. The Model

Consider a profit-maximizing retailer who seeks to allocate her shelf space capacity to n products. We are interested in characterizing the wholesale prices that will be quoted by the suppliers. Lower wholesale prices lead to larger shelf space but reduce the suppliers' unit profit margins. We model this situation as a sequential game, in which the suppliers play the role of leaders and the retailer plays the role of the follower. The timing of the game is the following: first, the suppliers set their wholesale prices, simultaneously, and then, the retailer chooses the shelf space allocation. We assume that all relevant cost and demand information is common knowledge. We solve the game backwards, by first solving the retailer's shelf space allocation problem and then analyzing the suppliers' decisions.

2.1. Demand Model

Let s_i be the amount of shelf space allocated to product i. Similarly to Corstjens and Doyle (1981), we assume that the demand for product i is an increasing concave function of the number of displays of product i. More products on the shelf lead to more demand, but the marginal returns of displaying a product are decreasing.

To highlight the effects of competition for shelf space, we assume that the demand for product i depends only on its shelf space s_i. We initially ignore the effects of marketing tactics such as retailer pricing and supplier advertising in our basic model, but investigate the impact of retail prices in §3.4. We also ignore the impact of the precise location on the shelf (e.g., products positioned at eye level seem to generate larger sales, see Dreze et al. 1994), and only consider the total space allocated to the product.

We assume, for simplicity, that all products share the same elasticity to shelf space. This assumption is not too restrictive if we consider products within the same category. In addition, most our results (existence and uniqueness of equilibrium) hold when products have different elasticities. We also ignore cross-elasticities among products, i.e., the dependency of the sales of product j from product i shelf space, as they are often referred to as "secondary effects" (Dreze et al. 1994).

Under these assumptions, demand for product i can be modeled as $a_i s_i^b$, where $a_i > 0$ is a scale parameter (depending on the brand of supplier i, its advertising policy, etc.) and b, $0 < b < 1$, is the shelf space elasticity. When $b \approx 0$, demand is insensitive to shelf space; in contrast, when $b \approx 1$, sales are directly proportional to space.

2.2. The Retailer's Allocation of Shelf Space

We model the retailer's problem as follows. We assume (without loss of generality) that the retailer has 1 unit of capacity that she seeks to allocate among n different products to maximize her profits. Let s_i be the amount of shelf space granted to product i. Hence, $\sum_{i=1}^{n} s_i \leq 1$. For tractability, we assume that s_i is a continuous variable.

Because shelf space allocation is a strategic decision (assortments are changed unfrequently and planograms are usually revised at most every couple of months), we consider a single-period model. Our model also ignores operational issues, such as day-to-day inventory replenishment. (In fact, store replenishment decisions are often made by suppliers, e.g., through Vendor-Managed Inventories, and have therefore limited impact on the retailer's allocation decision.) We finally neglect constraints on product availability, or required minimum or maximum shelf space allocations (Corstjens and Doyle 1983). While these constraints can easily be incorporated into our model, they unnecessarily complicate the analysis.

We then assume that the only relevant costs for the retailer are the products' gross profit margin. For simplicity, we assume linear costs; hence the profit margin for product i can be expressed as $r_i - w_i$, where r_i is the unit retail selling price, minus the inventory and handling costs, and w_i is the unit wholesale price for product i. Under these assumptions, the retailer seeks to allocate her shelf space so as to maximize her profits, that is,

$$\begin{aligned} \max \Pi_R = \sum_{i=1}^{n} (r_i - w_i) a_i s_i^b \\ \text{s. t.} \sum_{i=1}^{n} s_i \leq 1, \\ s_i \geq 0, \ \forall i. \end{aligned} \tag{1}$$

Define, for $i = 1, \ldots, n$, the gross margin of product i, i.e., the maximum profit that the retailer can obtain with product i:

$$m_i := a_i (r_i - w_i). \tag{2}$$

The optimal space allocation is such that

$$s_i = \frac{m_i^{\frac{1}{1-b}}}{\sum_{j=1}^{n} m_j^{\frac{1}{1-b}}}. \tag{3}$$

Under this allocation scheme, all products are somewhat complementary. Indeed, given the limited shelf space, the (continuous) solution for $b < 1$ is always to provide some space for each product. Only when $b \approx 1$, i.e., when demand is strongly sensitive to the number of displays, will the retailer allocate the entire shelf space to the product with the highest margin m_i.

Interestingly, Equation (3) yields that

$$\frac{s_i}{s_j} = \left(\frac{m_i}{m_j} \right)^{\frac{1}{1-b}} \tag{4}$$

which implies that the relative space for supplier i over supplier j depends only on the net margin ratio $\dfrac{m_i}{m_j} = \dfrac{a_i(r_i - w_i)}{a_j(r_j - w_j)}$.

2.3. Suppliers' Pricing Strategies

Each product is procured from a distinct vendor. While the retailer maximizes profits over the entire product category, suppliers are only concerned about the profit from their own products. The pricing decision is strategic, because it affects the retailer's shelf space allocation. We therefore ignore short-term inventory considerations (e.g., quantity discounts) in our model, because of their limited impact on the allocation decision. With linear production costs c_i and with a wholesale-price contract w_i, supplier i's gross profit margin equals $w_i - c_i$.

The suppliers' pricing decisions need to take into account the competitors' prices, because they influence the space allocation. Specifically, since we model the retailer as a follower, each supplier anticipates the retailer's space allocation given the competitors' wholesale prices. Hence, supplier i's problem can be formally stated as

$$\max \Pi_{Si} = (w_i - c_i) a_i \Big[s_i(w_1, \ldots, w_n) \Big]^b. \tag{5}$$

The demand function for supplier i, $a_i\Big[s_i(w_1,\ldots,w_n)\Big]^b$, is decreasing with w_i, and increasing with w_j, $j \neq i$, which is a standard condition for substitute products. It does not, however, have increasing differences, in contrast to most competitive demand models (such as separable demand functions). The increasing difference property means that decreasing the price of any product results in a greater increase in the demand for that product for lower levels of the price of any other product (Topkis 1998). It is easy to show that the demand for supplier i has decreasing differences when $s_i \leq 50\%$ and increasing differences otherwise. Hence, with $n=2$, when supplier 2 has the largest shelf space share, the increase in supplier 1's demand resulting from a decrease in w_1 is larger when w_2 is higher, and not when w_2 is lower, as most models typically assume.

2.4. The Supply Chain Perspective

Let us consider the supply chain total profit,

$$\Pi_{SC} := \Pi_R + \sum_{i=1}^n \Pi_{Si} = \sum_{i=1}^n a_i(r_i - c_i)\Big[s_i(w_1,\ldots,w_n)\Big]^b \tag{6}$$

When suppliers quote $(w_1,\ldots,w_n)$, the allocation of space may be suboptimal for the supply chain. Indeed, the concentration of the shelf-space allocation in the hands of the retailer, coupled with the competition among suppliers, creates a negative externality, because the resulting space allocation may not maximize the total profit of the supply chain.

We consider as a benchmark the centralized (or integrated) supply chain, as if there were a single decision-maker operating the entire supply chain. We denote the maximum supply chain profit associated with product i by

$$m_i^* := a_i(r_i - c_i), \tag{7}$$

and the supply-chain optimal allocation of space is

$$s_i^* = \frac{(m_i^*)^{\frac{1}{1-b}}}{\sum_{j=1}^n (m_j^*)^{\frac{1}{1-b}}}. \tag{8}$$

The corresponding supply chain profit is equal to

$$\Pi_{SC}^* = \sum_{i=1}^n (r_i - c_i) a_i \Big(s_i^*\Big)^b = \left[\sum_{i=1}^n (m_i^*)^{\frac{1}{1-b}}\right]^{1-b}. \tag{9}$$

In the sequel, we measure supply chain efficiency as the ratio of the integrated supply chain profit to the decentralized supply chain profit, that is,

$$\frac{\Pi_{SC}^*}{\Pi_{SC}} = \frac{\left[\sum_{i=1}^n (m_i^*)^{\frac{1}{1-b}}\right]^{1-b} \left[\sum_{i=1}^n m_i^{\frac{1}{1-b}}\right]^b}{\sum_{i=1}^n m_i^* m_i^{\frac{b}{1-b}}}. \tag{10}$$

In particular, we use the Price of Anarchy (PoA), defined as the maximum (or supremum) ratio of profits between the centralized supply chain and the decentralized supply chain, among all possible problem instances, i.e., parameters, $\{a_i\}_i, \{r_i\}_i, \{c_i\}_i$, and b. Because global optimization dominates sequential optimization, PoA is always greater than or equal to one.

3. Space Allocations in Equilibrium

In this section, we characterize the Nash equilibrium wholesale prices and space allocation, i.e., the pure strategy $(w_1^e, \ldots, w_n^e)$ from which no supplier has incentive to unilaterally deviate. We study the sensitivity of the results to the model parameters, and quantify supply chain efficiency.

3.1. Existence and Uniqueness of Equilibrium

We first show that there exists a unique equilibrium to the decentralized game.

THEOREM 1. *The game with n players has a unique pure strategy Nash equilibrium $(w_1^e, \ldots, w_n^e)$, characterized by the following conditions: for $i = 1, \ldots, n$,*

$$\frac{r_i - w_i}{r_i - c_i} = \frac{b - bs_i}{1 - bs_i}. \tag{11}$$

In particular, with two suppliers, the unique Nash equilibrium (w_1^e, w_2^e) is defined by the following conditions:

$$\frac{r_1 - w_1}{r_1 - c_1} = \frac{bs_2}{1 - bs_1}, \tag{12}$$

$$\frac{r_2 - w_2}{r_2 - c_2} = \frac{bs_1}{1 - bs_2}. \tag{13}$$

Thus, the percentage of the total margin captured by supplier i, $(r_i - w_i)/(r_i - c_i)$, should be set equal to a function of the space, $bs_j/(1 - bs_i)$. As a result, the best response function of supplier i, $w_i^{b.r.}(w_j)$, is increasing in w_j.

A side result of the theorem is that the shelf space allocation tends to be more even in the decentralized channel than in the integrated channel.

PROPOSITION 1. *$s_i^* \geq s_j^*$ if and only if $s_i^e \geq s_j^e$. In addition, if $s_i^* \geq s_j^*$ then $1 \leq \frac{s_i^e}{s_j^e} \leq \frac{s_i^*}{s_j^*}$; otherwise, $1 \geq \frac{s_i^e}{s_j^e} \geq \frac{s_i^*}{s_j^*}$.*

Therefore, suppliers' competition distorts the value of the wholesale prices, relative to the value of the unit production costs, resulting in a suboptimal shelf space allocation. Specifically, the least attractive products are given too much space, to the detriment of the most attractive products. Nevertheless, the order of shelf space allocations is preserved under decentralized decision-making, i.e., $s_i^e \geq s_j^e$ when $s_i^* \geq s_j^*$.

3.2. Sensitivity Analysis

We next investigate how wholesale prices, shelf space allocations, and profits for the suppliers, the retailer, and the whole supply chain vary when the problem parameters change.

PROPOSITION 2. *The equilibrium wholesale prices, allocated space and profits are such that*

(a) w_i^e *is increasing with* c_i, r_i *and* a_i*;*

(b) w_j^e, $j \neq i$, *is increasing with* c_i *and decreasing with* r_i *and* a_i*;*

(c) s_i^e *is decreasing with* c_i *and increasing with* r_i *and* a_i*;*

(d) s_j^e, $j \neq i$, *is increasing with* c_i *and decreasing with* r_i *and* a_i*;*

(e) Π_{Si}^e *is decreasing with* c_i *and increasing with* r_i *and* a_i*;*

(f) Π_{Sj}^e, $j \neq i$, *is increasing with* c_i *and decreasing with* r_i *and* a_i*;*

(g) Π_R^e *is decreasing with* c_i *and increasing with* r_i *and* a_i*;*

(h) and Π_{SC}^e *is quasi-convex (increasing or decreasing) in* c_i, r_i *and* a_i*.*

Wholesale prices always increase with costs. The supplier who suffers from the cost increase therefore obtains smaller space allocation and lower profits. In contrast, the competing suppliers take advantage of their dominant position by obtaining larger shelf space while charging higher wholesale prices. As a result, the retailer's profit decreases with the suppliers' costs. It is therefore in the retailer's interest to participate to cost-reduction programs at its suppliers (e.g., Wal-Mart Stores, Inc. has invested a lot of efforts to cut packaging waste at its suppliers, see Kabel 2007), because it leads to a reduction in wholesale prices, not only from the suppliers involved in the program, but also from their competitors.

Also, the suppliers' wholesale prices, allocated spaces, and profits increase with the final price of their products r_i and decrease with the prices of their competitors' products r_j, $j \neq i$. On the other hand, the retailer's profit always increases after a retail price rise. Similarly, when the market size of a given supplier a_i increases, its wholesale price, allocated space and profits increase, while they decrease for the competing suppliers, and the retailer's profit increases. Thus, suppliers' marketing efforts for increasing the brand awareness of their products, allowing them to increase the retail prices of their products or to expand the size of their market, not only benefit them, as well as the retailer, but also harm their competitors. Despite the decreasing marginal returns of space on demand, competition for shelf space can almost be seen as a zero-sum game, where any gain by one supplier is counterbalanced by a loss by the other suppliers.

Interestingly, the retailer also benefits from strengthened brand names. Therefore, supply chain-wide efforts can be devoted to increasing the strength of a brand, as all parties may gain from the resulting increase in revenue. In particular, retailers have become extremely powerful at helping build strong brand names (mindspace) with their advertising, promotions, and displays (shelfspace). As proposed by Corstjens and Corstjens (1995), "Shelfspace and mindspace are linked and complementary. If a product has achieved considerable mindspace–if it is present and liked in many consumer minds–this in itself will be a powerful incentive for the distribution to stock it. On the other hand, shelfspace is a powerful generator of mindspace. Seeing a product regularly helps increase its presence in the consumer's mind, and improves its image by suggesting it is popular."

From (h), the supply chain profit is quasi-convex in the cost, selling price, and market size. Hence, a cost increase may create a positive externality on the supply chain, when the cost is large. Intuitively, the supply chain profits improve when the supplier who experiences the cost increase is also the most expensive. From Proposition 1, this supplier receives a larger space allocation than what would have be optimal for the integrated supply chain. This above-optimal shelf space share exerts pressure puts the other suppliers under pressure, and results in an increase in the supply chain total profit. Alternatively, a cost reduction program may not always be beneficial to the entire supply chain. Similarly to the effects of changes in costs, an increase in the selling price r_i or the market size a_i may induce a negative externality on the supply chain profits.

Finally, for completeness, we investigate the impact of the sales elasticity with respect to shelf space, that is, b, on the wholesale prices, shelf space allocation, and profits. In contrast to the changes in unit production costs, selling prices and market sizes, changes in elasticity lead to non-monotonic effects. Specifically, wholesale prices are non-monotonic functions of the sales space elasticity, leading to non-monotonic behavior of the profit functions. Figure 1 illustrates the non-monotonic behavior of wholesale prices and profits as a function of elasticity b.

The next proposition characterizes how the shelf space allocation changes with b. When demand becomes more sensitive to the number of displays, the most attractive products (from the retailer's standpoint) receive more facings, to the detriment of the least attractive products.

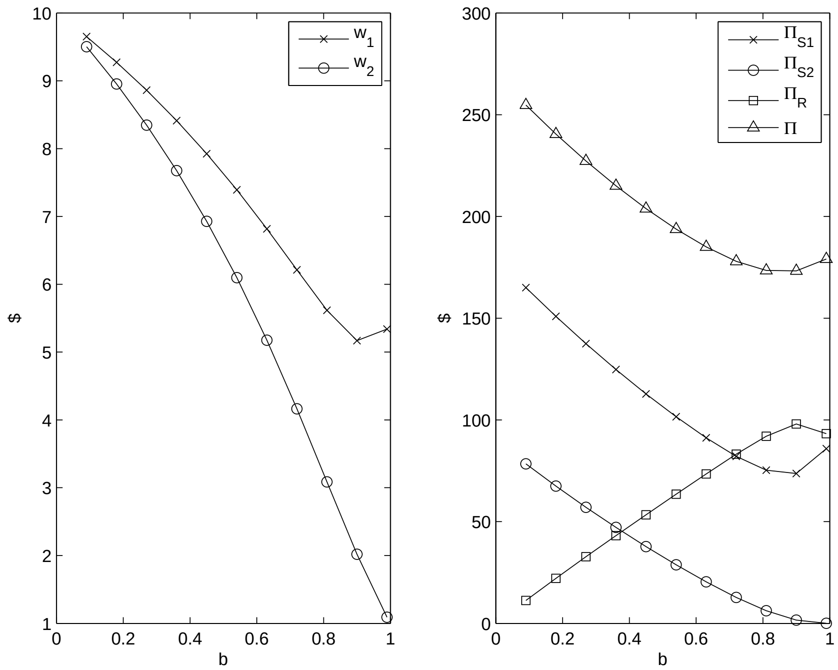
Proposition 3. *Without loss of generality, assume that $s_1^* \geq \ldots \geq s_n^*$. Then, there exists $k \in [1, \ldots, n]$ such that, for $i \leq k$, $\frac{ds_i^e}{db} \geq 0$ and for $i > k$, $\frac{ds_i^e}{db} \leq 0$.*

For $n = 2$, Proposition 3 implies that the supplier with the larger s_i^* captures more shelf space as b increases, to the expense of the supplier with the lower s_i^*.

3.3. Supply Chain Efficiency

In this section, we characterize the loss of efficiency resulting from decentralizing the decision making in the supply chain. We first analyze the basic model introduced in §2 and then consider

Figure 1 Comparative statics with changes in elasticity



Note. The data for this example are the following: $n=2$, $c_1=c_2=1$, $r_1=r_2=10$, $a_1=20$ and $a_2=10$ (and thus $1 \le \frac{s_1^e}{s_2^e} \le \frac{s_1^*}{s_2^*}$ from Proposition 1).

what happens when the retailer can make pricing decisions as well (§3.4), and when the retailer pursues different objectives than profit maximization (§3.5). We use PoA to measure supply chain efficiency, defined as the maximum ratio (10) over all problem instances.

THEOREM 2. *With n suppliers, the Price of Anarchy is equal to the following maximum*

$$PoA_n = \max_{s_1,\ldots,s_n \ge 0, 0\le b\le 1} \frac{\left[\sum_{i=1}^n s_i \left(\frac{1-bs_i}{1-s_i}\right)^{\frac{1}{1-b}}\right]^{1-b}}{\sum_{i=1}^n s_i \left(\frac{1-bs_i}{1-s_i}\right)} \text{ subject to } \sum_{i=1}^n s_i = 1. \tag{14}$$

COROLLARY 1. *The Price of Anarchy is increasing in n, i.e.,* $PoA_{n+1} \ge PoA_n$.

Corollary 1 follows from (14), where, for the case of $n+1$ suppliers, we set $s_{n+1}=0$. As a result the PoA with $n+1$ suppliers must be greater than or equal to that with n. Given that PoA_n is increasing in n, we can solve the optimization problem for $n=2$ and $n=\infty$, to provide bounds.

In addition, Lemma 1 in appendix demonstrates that, for any given n, the n-variable optimization problem in the right-hand side of (14) can be simplified into a two-variable optimization problem. Therefore, the PoA can be computed numerically, by solving a two-variable optimization problem.

The next two propositions show that that the PoA is relatively low and insensitive to the number of suppliers.

PROPOSITION 4. *The Price of Anarchy for $n=2$ is $PoA_2 \in [1.051, 1.052]$.*

PROPOSITION 5. *The Price of Anarchy for $n=\infty$ is $PoA_\infty \in [1.055, 1.056]$.*

A loss of efficiency of 5-6% might seem low, especially when compared to the PoA bounds derived in supply games with stochastic demand (i.e., 4/3 in supply games with option contracts, see Martínez-de-Albéniz and Simchi-Levi 2003; and $e-1$ with wholesale-price contracts, see Perakis and Roels 2007). This small level of inefficiency can be explained by the presence of an alternative use of capacity. In traditional supply games, unused capacity is lost, and the reservation profit of the follower in those games is often set to zero (with the exception of Lariviere and Porteus 2001 and Bernstein and Marx 2006). In contrast, in the shelf-space allocation game, all capacity is utilized, softening the impact of suboptimal decisions. Nevertheless, profit margins are thin in retail, and a 5% increase in efficiency can make a real impact on the bottom line.

3.4. When the Retailer Takes Pricing Decisions

We now investigate the impact of letting the retailer choose the selling prices to the end-consumers. Somewhat surprisingly, supply chain efficiency decreases, despite the fact that the retailer has now more levers to coordinate the channel. In fact, the retailer can make sub-optimal decisions (from a supply chain standpoint) not only in shelf space allocation, but also in pricing.

For this purpose we consider a variant of Martín-Herrán et al. (2006), where the customer demand for i is equal to $a_i r_i^{-\mu} s_i^b$, with $\mu \geq 1$. The demand for product i is thus a decreasing convex function of the retail prices r_i, and all products share the same price elasticity μ. A price increase of product i has no direct effect on the demand of other products, but indirectly influences the shelf space allocation.

If the supply chain were integrated, the optimal pricing scheme is $r_i(c_i) = \mu c_i/(\mu - 1)$ (with a slight abuse of notation, by defining $r_i(.)$ as a function). In a decentralized channel however, prices are set equal to $r_i(w_i) = \mu w_i/(\mu - 1)$ and are therefore larger. The retailer's optimal space allocation follows (3) with $m_i = a_i (r_i(w_i) - w_i)(r_i(w_i))^{-\mu} = (\mu-1)^{\mu-1}/\mu^\mu a_i w_i^{-(\mu-1)}$. The shelf space allocated to product i is therefore decreasing with w_i, similarly to the basic model introduced in §2. We derive an equilibrium result, analogous to Theorem 1.

THEOREM 3. *When the retailer sets the prices r_i in addition to allocating the space s_i, then the* *game with n players has a unique pure strategy Nash equilibrium $(w_1, \ldots, w_n)$, characterized by the following conditions: for $i = 1, \ldots, n$,*

$$\frac{w_i}{c_i} = 1 + \frac{1-b}{(\mu-1)(1-bs_i)}. \tag{15}$$

The next Proposition quantifies the PoA when pricing decisions are endogenous and strikingly contrasts with Propositions 4 and 5.

Proposition 6. *When the retailer sets prices optimally, $PoA = e/2$, where e is the exponential number, i.e., $e = 2.7182...$.*

Consequently, the decentralized supply chain may be very inefficient when pricing decisions are endogenous. Intuitively, the supply chain is inefficient because of double marginalization (Spengler 1950), not because of competition for shelf space. In fact, the worst-case problem instance characterizing the PoA bound, used in the proof of Proposition 6, is independent of the space allocation.

The bound $e/2$ is remarkably close to the bounds derived previously, in other double-marginalization games: 4/3 with option contracts (Martínez-de-Albéniz and Simchi-Levi 2003), and $e-1$ with wholesale-price contracts (Perakis and Roels 2007). Based on this observation, we conjecture that double marginalization can generate about 25-40% inefficiencies in the presence of decreasing marginal returns (either through a concave demand-price relationship or through a demand probability distribution). Larger bounds can obviously be derived when marginal returns are nondecreasing, such as Cournot-based competition models.

Comparing Proposition 6 with Propositions 4 and 5 also reveals that the inefficiencies arising from shelf space competition much smaller than those arising from double marginalization. Nevertheless, shelf space competition generates inefficiencies on its own, and is a significant issue given the thin margins in retail.

3.5. Other Space Allocation Rules

So far, we have assumed that the retailer allocated her limited shelf space to maximize her gross profits. In particular, the optimal shelf space allocation was assumed to be based on the gross profit margin contribution $m_i = a_i(r_i - w_i)$ of each supplier.

Many commercial software programs are however based on different allocation rules, such as sales, sales per square foot, gross profit margin per square foot, or stocking expense (see Curhan 1973 and Corstjens and Doyle 1983 for a review). These alternate objectives are not necessarily irrational or suboptimal, because many practical issues (e.g., generating store traffic) are ignored in our idealized model. It is therefore critical to understand whether supply chain efficiency improves or deteriorates under these alternate objectives.

Allocation based on gross profit margin per square foot. When the retailer allocates the shelf space proportionally to the products' profit margins per square foot, supply chain efficiency decreases. Indeed, based on this rule, the shelf space allocation satisfies

$$s_i = \frac{a_i(r_i - w_i)}{\sum_{j=1}^{n} a_j(r_j - w_j)}$$

Anticipating this allocation rule, the suppliers will set their wholesale prices to maximize their profits. The next proposition characterizes the Price of Anarchy in this case.

Proposition 7. *When the retailer bases the shelf-space allocation on gross profit margin per square foot instead of gross profit margin, the Price of Anarchy is larger.*

Therefore, by committing to a suboptimal allocation rule, the retailer worsens the efficiency of the channel. As a matter of fact, we find that $PoA_2^{\$/sqft} \in [1.298, 1.299]$.

Allocation based on sales. When the retailer allocates the shelf space proportionally to sales or revenue, possibly divided by the square footage, the allocation decision is independent from the wholesale prices. In this case, the supplier profit functions are increasing with their respective wholesale prices, and it is optimal for them to charge the highest possible wholesale price, i.e., $w_i = r_i$ for all i. As a result, the retailer's profit is equal to zero. Supply chain efficiency might however increase, depending on the value of the parameters.

4. Retailing Practices: Towards a More Efficient Supply Chain?

In this section, we analyze different retailing practices and discuss their impact on the shelf allocation game. In particular, we are interested in whether they help improve supply chain efficiency.

We first discuss the potential of *pay-to-stay* fees, a rent that retailers charge manufacturers for space. We show that these contracts can coordinate the channel. Second, we analyze the impact of horizontal integration (i.e., when a manufacturer owns several brands) and vertical integration (e.g., when a retailer sells private-label brands). Vertical integration can also be viewed as the widespread—and controversial—practice of *category management*, according to which the retailer delegates the management of the category to one of the suppliers, called the category captain.

4.1. Pay-to-Stay Fees

Suppliers often pay fees to ensure the continued presence of their product on the shelf for some period (commonly one year). These fees are usually called *pay-to-stay fees*. We show that pay-to-stay fee contracts always increase the retailer's profit, in comparison to a situation where only wholesale prices are contractible. Interestingly, these fees are never profitable to the suppliers.

Thus, we assume that the shelf space is contractible, and that it is sold through an auction. At equilibrium, all suppliers pay the same amount per unit of shelf space, denoted by f, and the sum of requested shelf spaces equals the total shelf capacity. The retailer's profit is then equal to the sum of the profit from sales and the additional revenue from the pay-to-stay fee collection, that is,

$$\Pi_R = \sum_{i=1}^{n} (r_i - w_i^{PTS}) a_i \left(s_i^{PTS}\right)^b + f,$$

where superscript PTS refers to "pay-to-stay" fees. Here, s_i^{PTS} is determined by the suppliers, that pay for each unit of space a price f. In contrast to the model introduced in §2, the retailer does not make any decision here since both the pay-to-stay fee and the shelf space allocation are negotiated among the suppliers.

The suppliers' revenue is the profit from sales less the pay-to-stay fee. For a given fee f, all suppliers need to decide their wholesale prices and the amount of shelf space capacity they want to be allocated. The equilibrium fee f then balances supply with demand: if supplier i requests s_i^{PTS} units of capacity, the equilibrium fee f is such that $\sum_{i=1}^{n} s_i^{PTS} = 1$. Plugging this equilibrium condition into the suppliers' profit functions leads to the following game:

$$\max_{w_i, s_i} \Pi_{Si}(w_i, s_i) = (w_i - c_i) a_i s_i^b - f s_i, \ \forall i.$$

The first-order optimality conditions for each supplier i, which are necessary and sufficient (because the profit functions are concave), are given by

$$\frac{\partial \Pi_{Si}}{\partial w_i} = a_i s_i^b > 0,$$
$$\frac{\partial \Pi_{Si}}{\partial s_i} = (w_i - c_i) b s_i^{b-1} a_i - f = 0.$$

It is thus optimal to set $w_i^{PTS} = r_i$, and to allocate the shelf space capacity so that $f = a_i(r_i - c_i) b \left(s_i^{PTS}\right)^{b-1}$ for all i. The latter condition implies that $s_i^{PTS} = s_i^*$ for all i; consequently, the shelf space allocation with pay-to-stay fees is supply-chain optimal, and because the total supply chain profits only depend on the shelf space allocation, $PoA^{PTS} = 1$. The equilibrium pay-to-stay fee equals

$$f = \frac{b m_i^*}{\left(s_i^*\right)^{1-b}} = b \left[\sum_{k=1}^{n} \left(m_k^*\right)^{\frac{1}{1-b}} \right]^{1-b}.$$

The next proposition compares the profits with and without pay-to-stay fees. While the supply chain is globally efficient in the presence of these fees, no supplier earns additional profits. Because only the retailer benefits from this contractual arrangement, it is likely that the initiative of imposing pay-to-stay fees and transferring all the profit margin from sales to the suppliers will originate from the retailer in practice.

PROPOSITION 8. *By switching from wholesale-price contracts to pay-to-stay fees, (a) all the suppliers earn lower profits, and (b) the retailer earns higher profits.*

With pay-to-stay fees, the retailer's total profit comes from the fees, and not from the sales revenue. Therefore, under this contract, the retailer is only rewarded for her core capability of warehousing and shelf-space leaser. Brown and Tucker (1961) and Cairns (1962) already suggested that suppliers should pay retailers for their desired shelf space, and that these payments should exceed the retailers' opportunity costs for using such space. Interestingly, Cairns (1962) proposed that the price offered by a supplier for a unit of shelf space should be equal to the product of the retailer's unit profit margin with the ratio of sales to space. In comparison, our model suggests that the price f should be equal to the supply chain's unit profit margin $r_i - c_i$, multiplied by the ratio of sales to space, $a_i\left(s_i^*\right)^{b-1}$, and weighted by the sales-space elasticity b. Hence, our model is consistent with Cairns' argument, but refines his proposal by considering the entire supply chain's, instead of the retailer's, profit margin.

In practice, however, retailers are more than shelf-space leasers, because of their ability to influence the sales of particular product (e.g., through advertising) and they should be rewarded for this function as well. In fact, retailers have recently become more powerful, by exploiting the value of their contact with the consumer and realizing the importance of the marketing variables (price, display, promotion) under their control. Shelf-space lease is certainly an important operational lever for retailers, but it needs to be aligned with their strategy for influencing consumer choice and gaining mindspace (Corstjens and Corstjens 1995).

4.2. Integrating the Supply Chain

Horizontal Integration. If suppliers are horizontally integrated, i.e., one supplier owns all the products in the shelf, full efficiency can be achieved as follows. The supplier, acting as a leader, sets the wholesale prices equal to the retail prices. Because she earns zero margin, the retailer is indifferent about the shelf space allocation and can be encouraged to choose the allocation that maximizes the supply chain total profits (possibly by being offered wholesale prices slightly below the retail prices). Therefore, the incentive misalignment disappears as soon as the suppliers are horizontally integrated.

This result sheds light on the brand strategy of large consumer good producers such as Procter & Gamble or Unilever: even if the products are competing for the same shelf space, and are somehow cannibalizing the sales of each other, they give the consumer-good producer enough power to control the shelf-space allocation and capture at the same time significant profit margins.

Vertical Integration. We now analyze a model of vertical integration, where the retailer is vertically integrated with one of the suppliers. This model has two different interpretations.

There is vertical integration when a retailer owns one of the brands, i.e., when one of the products is a private-label brand. The effects of vertical integration are becoming more important as brands introduced by retailers (such as Wal-Mart Stores, Inc. and Target Corporation) knock many second-tier brands off the shelves, reducing the product category to a few brands next to their own private-label brands (Jubak 2005).

This model of vertical integration is also representative of category management. Category management is a marketing initiative that recommends centrally managing the entire product category instead of managing each brand in a decentralized fashion. Ideally, category management should be adopted by all supply chain partners, i.e., both the retailers and suppliers, to make globally optimal decisions and obtain superior profits. In practice however, the retailer appoints one of her suppliers, called the "category captain," to manage the entire product category on her behalf. Indeed, suppliers have typically more information than the retailers about product costs, planned promotions and new product introductions, as well as the end-consumer demand (since they have an aggregated view of the market). However, as we shall see, the category captain is also biased towards increasing the sales of his own product, and will be tempted to push his product to the detriment of the other suppliers' products.

Let us assume that supplier 1 is the category captain. To keep the analysis general, we do not model the specific terms of agreement between the retailer and the category captain. Indeed, mutual trust is generally considered as a prerequisite to the success of category management, see Steiner (2001). Instead, we consider the retailer and the category captain as being integrated into a single firm. Using the same game as in the decentralized setting, we assume that suppliers i, $i = 2, \ldots, n$, first decide their wholesale prices w_i^{CM} (where superscript CM refers to category management), and then the retailer and supplier 1 jointly decide the shelf space allocation. Formally, we let $w_1^{CM} = c_1$. The retailer's problem is therefore to choose the shelf space allocation $s_1^{CM}, \ldots, s_n^{CM}$

that maximizes the joint profit $(r_1 - c_1)a_1 s_1^b + \sum_{i=2}^{n}(r_i - w_i^{CM})a_i s_i^b$. Similar to the games analyzed before, it is optimal to allocate space following Equation (3).

The next proposition analyzes the changes in supplier i's wholesale price and profit, $i = 2, \ldots, n$, as well as the total supply chain profit, with category management.

PROPOSITION 9. *With category management, for* $i = 2, \ldots, n$,

(a) $w_i^{CM} \leq w_i^e$;

(b) $s_i^{CM} \leq s_i^e$;

(c) $\Pi_{Si}^{CM} \leq \Pi_{Si}^e$;

(d) and $\Pi_{S1}^{CM} + \Pi_R^{CM} \geq \Pi_{S1}^e + \Pi_R^e$.

Therefore, vertical integration always hurts the suppliers who are not the category captain, even though they still act as Stackelberg leaders. In particular, suppliers excluded from the coalition are allocated less shelf space despite their lower wholesale prices. Using a different model, in which demand is insensitive to shelf space and the retailer chooses the selling prices, Kurtuluş and Toktay (2005) also find that category management is beneficial to the retailer and the category captain, and harmful to the excluded suppliers. Our model therefore corroborates their conclusions when retail prices are fixed.

In fact, the collusion might be so harmful to the excluded suppliers that the total supply chain efficiency might even decrease. The next proposition computes the Price of Anarchy of a vertically integrated supply chain, and shows that it is larger than the PoA of a completely decentralized channel (Proposition 4).

PROPOSITION 10. *With category management and* $n = 2$, $PoA_2^{CM} \in [1.079, 1.080]$.

In fact, the Price of Anarchy is maximized when $m_1^* = m_2^*$, i.e., $a_1(r_1 - c_1) = a_2(r_2 - c_2)$. At this point, not only $s_1^* = s_2^*$, but also $s_1^e = s_2^e$ in the basic model (from Proposition 1). Hence, the same shelf space allocation can achieve full efficiency of a completely decentralized supply chain, while at the same time be associated with the worst efficiency of a vertically integrated chain under category management. Moreover, vertical integration is the least effective at improving channel efficiency when the two products have comparable profit rates.

The worst-case performance of vertical integration, relative to that of a decentralized supply chain, sheds light on one of the main pitfalls of category management. The goal of category management is to improve channel efficiency, by centrally managing the product category, so that each partner is better off. In reality, category management raises antitrust concerns because it leads to noncompetitive coalitions, favoring one supplier over another (Steiner 2001, Bush and Gelb 2005). Our model of vertical integration shows that if one of the partners is left aside from the coalition, her margins and sales will plummet, as well as her profit, while channel efficiency might be even worse than without category management. As an illustration, a manager reported in a Federal Trade Commission workshop panel that "the competitor was able to reduce my shelf space to I call it unlivable living conditions and unlivable space" (FTC Report 2001). Consequently, the antitrust concerns about category management are well grounded, especially given that the practice worsens the overall supply chain efficiency (without mentioning the detrimental impact it might have on the end-consumer through higher prices and reduced variety).

5. Conclusions

This paper introduces a model of supply chain competition for shelf space. Our model builds on the shelf space allocation model by Corstjens and Doyle (1983) to analyze the competitive pressure on suppliers to obtain shelf space. When the retailer allocates her shelf space so as to maximize her profit (or the gross margin per square foot), suppliers can increase their space share by reducing their wholesale prices. We show the existence and uniqueness of an equilibrium in the wholesale pricing game between suppliers. The equilibrium prices of all suppliers are increasing with any supplier's cost; they are increasing on the corresponding selling price, but decreasing on a competitor's selling price.

We also characterize the loss of efficiency in the decentralized supply chain, using the Price of Anarchy. In particular, we demonstrate that the inefficiencies created in the space allocation process are minimal, less than 6%, with wholesale price contracts, and that full coordination is achieved with pay-to-stay fee contracts. On the other hand, coordination of retail prices is very necessary, as double marginalization may lead to a 30% loss of efficiency.

Finally, we examine the impact of some retailing practices on the space allocation decision, with a particular attention to category management. Specifically, we show that category management benefits the retailer and the category captain, and hurts the other suppliers, even when pricing decisions are exogenous, formalizing and quantifying the antitrust concerns against the practice.

The current model can be extended in several directions. First, one could consider the assortment size to be endogenous. In the current model, supply chain profits always improve as n increases. However, restricting the number of SKUs intensifies competition between suppliers, potentially leading to larger retailer's profits. Thus, if the assortment decision is made by the retailer, the optimal assortment size will result from the trade-off between the increase in supply chain profit (the size of the pie) and the pressure put on the suppliers (the share for the retailer). Another promising extension would be to model inventory decisions with stochastic demand. Conceivably, the performance of supply contracts (e.g., wholesale-price, buyback, quantity discount) is significantly affected by competition for shelf space, and it would be interesting to assess how the previous conclusions about their coordination potential (see Cachon 2003) carry over in a multiple product environment.

Appendix

Proof of Theorem 1

Proof. The first derivative of supplier i's profit function is equal to

$$\frac{d\Pi_{Si}}{dw_i} = a_i s_i^b + a_i(w_i - c_i) b s_i^{b-1} \frac{ds_i}{dw_i} = \frac{a_i s_i^b (w_i - c_i)}{r_i - w_i} \left(\frac{r_i - w_i}{w_i - c_i} + \frac{b(r_i - w_i)}{s_i} \frac{ds_i}{dw_i} \right).$$

Noting that

$$\frac{1}{s_i}\frac{ds_i}{dw_i} = -\frac{1 - s_i}{(1-b)(r_i - w_i)},$$

$\dfrac{r_i - w_i}{w_i - c_i} + \dfrac{b(r_i - w_i)}{s_i}\dfrac{ds_i}{dw_i}$ is decreasing with w_i, and thus Π_{Si} is quasi-concave in w_i. The best-response function is thus well-defined by $\dfrac{r_i - w_i}{w_i - c_i} = \dfrac{b(1 - s_i)}{1 - b}$ or equivalently

$$\frac{r_i - w_i}{r_i - c_i} = \frac{b(1 - s_i)}{1 - b s_i}.$$

Because each supplier's profit function is continuous quasi-concave in w_i, and that the strategy space is the compact convex interval $[c_1, r_1] \times \ldots \times [c_n, r_n]$, there exists a Nash equilibrium.

We now prove that the Nash equilibrium is unique. Using (11), the best-response function $m_i^{b.r.}(m_{-i})$ can be expressed as

$$\frac{m_i^{b.r.}}{m_i^*} = \frac{b(1 - s_i)}{1 - b s_i}.$$

We can log-differentiate implicitly with respect to m_j, and use that $\dfrac{1}{s_i}\dfrac{ds_i}{dm_i} = \dfrac{1 - s_i}{(1-b)m_i}$ and $\dfrac{1}{s_i}\dfrac{ds_i}{dm_j} = \dfrac{-s_j}{(1-b)m_j}$ to obtain

$$\frac{dm_i^{b.r.}}{dm_j} = -\frac{(1-b)m_i}{(1-s_i)(1-bs_i)} \left(\frac{ds_i}{dm_j} + \frac{ds_i}{dm_i}\frac{dm_i^{b.r.}}{dm_j} \right) = \frac{s_i s_j m_i}{(1 - s_i)\Big[1 + (1-b)s_i\Big] m_j} \geq 0. \tag{A-1}$$

Suppose that there are two equilibria $\mathbf{w}^{eq1}$ and $\mathbf{w}^{eq2}$. Without loss of generality, assume that $w_1^{eq1} \leq w_1^{eq2}$. From Equation (A-1), for all i, we must have $w_i^{eq1} \leq w_i^{eq2}$. Equation (11) imply that $s_i^{eq1} \geq s_i^{eq2}$. As $\sum_{i=1}^{n} s_i = 1$, all the inequalities are in fact equalities and hence $\mathbf{w}^{eq1} = \mathbf{w}^{eq2}$. ■

Proof of Proposition 1

Proof. Dividing the first-order optimality conditions (11) for i and j, it follows that

$$\frac{r_i - w_i}{r_j - w_j}\frac{r_j - c_j}{r_i - c_i} = \frac{1-s_i}{1-s_j}\frac{1-bs_j}{1-bs_i}. \tag{A-2}$$

If $s_i^e \geq s_j^e$, the above equality implies that $\frac{r_i - w_i}{r_j - w_j}\frac{r_j - c_j}{r_i - c_i} \leq 1$, and hence $1 \leq \frac{s_i^e}{s_j^e} \leq \frac{s_i^*}{s_j^*}$. The inequalities are reversed if $s_i^e \leq s_j^e$.

Thus, $s_1^* \geq \ldots \geq s_n^*$ if and only if $s_1^e \geq \ldots \geq s_n^e$. Any of the two statements implies that $\frac{s_1^e}{s_1^*} \leq \ldots \leq \frac{s_n^e}{s_n^*}$. ■

Proof of Proposition 2

Proof. The results are shown using the implicit function theorem and the chain rule. For this purpose, we consider then optimality equation (11), expressed through m_j, m_j^*, i.e., $\frac{m_j^e}{m_j^*} = \frac{b - bs_j^e}{1 - bs_j^e}$.

We take the log-derivative with respect to $m_i^* = a_i(r_i - c_i)$,

$$\frac{1}{m_i^e}\frac{dm_i^e}{dm_i^*} - \frac{1}{m_i^*} = -\frac{1-b}{(1-s_i^e)(1-bs_i^e)}\frac{ds_i^e}{dm_i^*} = -\frac{1-b}{(1-s_i^e)(1-bs_i^e)}\left(\sum_{k=1}^n \frac{ds_i}{dm_k}\frac{dm_k^e}{dm_i^*}\right),$$

and for $j \neq i$,

$$\frac{1}{m_j^e}\frac{dm_j^e}{dm_i^*} = -\frac{1-b}{(1-s_j^e)(1-bs_j^e)}\frac{ds_j^e}{dm_i^*} = -\frac{1-b}{(1-s_j^e)(1-bs_j^e)}\left(\sum_{k=1}^n \frac{ds_j}{dm_k}\frac{dm_k^e}{dm_i^*}\right). \tag{A-3}$$

Since $\frac{ds_i}{dm_i} = \frac{s_i(1-s_i)}{(1-b)m_i}$ and $\frac{ds_i}{dm_j} = \frac{-s_i s_j}{(1-b)m_j}$, we have

$$\begin{aligned}\frac{1}{m_i^e}\frac{dm_i^e}{dm_i^*} &= \frac{1}{m_i^*} - \frac{s_i^e}{(1-s_i^e)(1-bs_i^e)}\left(\sum_{k=1}^n \frac{-s_k^e}{m_k^e}\frac{dm_k^e}{dm_i^*} + \frac{1}{m_i^e}\frac{dm_i^e}{dm_i^*}\right)\\ &= \frac{\frac{(1-s_i^e)(1-bs_i^e)}{m_i^*} + s_i^e\left(\sum_{k=1}^n \frac{s_k^e}{m_k^e}\frac{dm_k^e}{dm_i^*}\right)}{(1-s_i^e)(1-bs_i^e) + s_i^e},\end{aligned}$$

and for $j \neq i$,

$$\begin{aligned}\frac{1}{m_j^e}\frac{dm_j^e}{dm_i^*} &= -\frac{s_j^e}{(1-s_j^e)(1-bs_j^e)}\left(\sum_{k=1}^n \frac{-s_k^e}{m_k^e}\frac{dm_k^e}{dm_i^*} + \frac{1}{m_j^e}\frac{dm_j^e}{dm_i^*}\right)\\ &= \frac{s_j^e\left(\sum_{k=1}^n \frac{s_k^e}{m_k^e}\frac{dm_k^e}{dm_i^*}\right)}{(1-s_j^e)(1-bs_j^e) + s_j^e}.\end{aligned} \tag{A-4}$$

A linear combination of the equations above yields that

$$\sum_{k=1}^n \frac{s_k^e}{m_k^e}\frac{dm_k^e}{dm_i^*} = \frac{\dfrac{s_i^e(1-s_i^e)(1-bs_i^e)}{m_i^*\left[(1-s_i^e)(1-bs_i^e) + s_i^e\right]}}{1 - \sum_{k=1}^n \dfrac{\left(s_k^e\right)^2}{(1-s_k^e)(1-bs_k^e) + s_k^e}} \geq 0 \tag{A-5}$$

and thus for all j,

$$\frac{dm_j^e}{dm_i^*} \geq 0. \tag{A-6}$$

In addition, $\dfrac{m_j^e}{m_j^*} = \dfrac{b - bs_j^e}{1 - bs_j^e}$ yields that for $j \neq i$,

$$\frac{ds_j^e}{dm_i^*} \leq 0 \text{ and hence } \frac{ds_i^e}{dm_i^*} \geq 0. \tag{A-7}$$

Interestingly, this implies that $\dfrac{1}{m_i^e}\dfrac{dm_i^e}{dm_i^*} - \dfrac{1}{m_i^*} \leq 0$ and hence

$$0 \leq \frac{dm_i^e}{dm_i^*} \leq \frac{m_i^e}{m_i^*} = \frac{b - bs_i^e}{1 - bs_i^e} \leq b \leq 1. \tag{A-8}$$

Noting that $\Pi_{Sj}^e = (m_j^* - m_j^e)\left(s_j^e\right)^b$ and $\Pi_R^e = \sum_{k=1}^n m_k^e \left(s_k^e\right)^b = \left[\sum_{k=1}^n \left(m_k^e\right)^{\frac{1}{1-b}}\right]^{1-b}$,

$$\frac{d\Pi_{Si}^e}{dm_i^*} = \left(1 - \frac{dm_i^e}{dm_i^*}\right)\left[s_i^e\right]^b + (m_i^* - m_i^e)b\left[s_i^e\right]^{b-1}\left(\frac{ds_i^e}{dm_i^*}\right) \geq 0, \tag{A-9}$$

for $j \neq i$,

$$\frac{d\Pi_{Sj}^e}{dm_i^*} = -\left(\frac{dm^e}{dm_i^*}\right)\left[s_j^e\right]^b + (m_j^* - m_j^e)b\left[s_j^e\right]^{b-1}\left(\frac{ds_j^e}{dm_i^*}\right) \leq 0, \tag{A-10}$$

and

$$\frac{d\Pi_R^e}{dm_i^*} \geq 0. \tag{A-11}$$

Finally, since $\sum_{k \neq i} \dfrac{ds_k}{dm_i^*} = -\dfrac{ds_i}{dm_i^*}$ and that $\left(\dfrac{s_i}{s_k}\right)^{1-b} = \dfrac{m_i}{m_k}$, and Equation (11), $\Pi_{SC}^e = \sum_{k=1}^n m_k^* \left(s_k^e\right)^b$ yields that

$$\begin{aligned}
\frac{d\Pi_{SC}^e}{dm_i^*} &= \left(s_i^e\right)^b + b\sum_{k=1}^n \frac{m_k^*}{\left(s_k^e\right)^{1-b}}\left(\frac{ds_k^e}{dm_i^*}\right) \\
&= \left(s_i^e\right)^b + b\sum_{k=1}^n \left(\frac{m_k^* m_i^e}{m_k^e \left(s_i^e\right)^{1-b}}\right)\left(\frac{ds_k^e}{dm_i^*}\right) \\
&= \left(s_i^e\right)^b + b\sum_{k \neq i} \left(\frac{m_k^* m_i^e}{m_k^e \left(s_i^e\right)^{1-b}} - \frac{m_i^*}{\left(s_i^e\right)^{1-b}}\right)\left(\frac{ds_k^e}{dm_i^*}\right) \\
&= \left(s_i^e\right)^{b-1}\left[s_i^e + bm_i^* \sum_{k \neq i}\left(\frac{m_k^* m_i^e}{m_k^e m_i^*} - 1\right)\left(\frac{ds_k^e}{dm_i^*}\right)\right] \\
&= \left(s_i^e\right)^{b-1}\left[s_i^e + bm_i^* \sum_{k \neq i}\left(\frac{1 - bs_k^e}{1 - s_k^e}\frac{1 - s_i^e}{1 - bs_i^e} - 1\right)\left(\frac{ds_k^e}{dm_i^*}\right)\right].
\end{aligned} \tag{A-12}$$

Let $z_k = \dfrac{s_k^e(1 - s_k^e)(1 - bs_k^e)}{s_k^e + (1 - s_k^e)(1 - bs_k^e)}$. Observing that

$$1 - \sum_{j=1}^n \frac{\left(s_j^e\right)^2}{(1 - s_j^e)(1 - bs_j^e) + s_j^e} = \sum_{j=1}^n \left(s_j^e - \frac{\left(s_j^e\right)^2}{(1 - s_j^e)(1 - bs_j^e) + s_j^e}\right) = \sum_{j=1}^n z_j,$$

and using Equations (A-3), (A-4) and (A-5), we have for $k \neq i$,

$$\frac{ds_k^e}{dm_i^*} = -\frac{(1 - s_k^e)(1 - bs_k^e)}{(1 - b)}\frac{1}{m_k^e}\frac{dm_k^e}{dm_i^*} = -\frac{z_i z_k}{(1 - b)m_i^* \sum_{j=1}^n z_j}.$$

Thus,

$$\begin{aligned}
\frac{d\Pi^e_{SC}}{dm^*_i} &= \left(s^e_i\right)^{b-1}\left[s^e_i - \frac{bz_i}{1-b}\sum_{k\neq i}\left(\frac{1-bs^e_k}{1-s^e_k}\frac{1-s^e_i}{1-bs^e_i}-1\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right)\right]\\
&= \frac{z_i(1-s^e_i)\left(s^e_i\right)^{b-1}}{(1-bs^e_i)}\left[\frac{s^e_i}{z_i}\frac{1-bs^e_i}{1-s^e_i} - \frac{b}{1-b}\sum_{k=1}^n\left(\frac{1-bs^e_k}{1-s^e_k}-\frac{1-bs^e_i}{1-s^e_i}\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right)\right]\\
&= \frac{z_i(1-s^e_i)\left(bs^e_i\right)^{b-1}}{(1-bs^e_i)}\left[\frac{s^e_i}{bz_i}\frac{1-bs^e_i}{1-s^e_i} + \sum_{k=1}^n\left(\frac{1}{1-s^e_i}-\frac{1}{1-s^e_k}\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right)\right].
\end{aligned}$$

We observe that

$$\frac{d}{ds_i}\left(\frac{s^e_i}{bz_i}\frac{1-bs^e_i}{1-s^e_i}\right)\geq 0$$

so it also increases with m^*_i. In addition, let $\theta_k = \frac{1}{1-s^e_i} - \frac{1}{1-s^e_k}$ and sort the suppliers so that $s^e_1 \geq \ldots \geq s^e_n$. Thus, $\theta_1 \leq \ldots \leq \theta_n$.

$$\begin{aligned}
\frac{d}{dm^*_i}\left(\frac{\sum_{k=1}^n \theta_k z_k}{\sum_{k=1}^n z_k}\right) &\geq \frac{\sum_{k=1}^n\sum_{j=1}^n\left(-\frac{1}{(1-s^e_k)^2}\right)\frac{ds^e_k}{dm^*_i}z_j z_k + \theta_k\left(\frac{dz_k}{ds^e_k}\frac{ds^e_k}{dm^*_i}z_j - z_k\frac{dz_j}{ds^e_j}\frac{ds^e_j}{dm^*_i}\right)}{\left(\sum_{k=1}^n z_k\right)^2}\\
&= \frac{z_i\sum_{k=1}^n\sum_{j=1}^n\left[\left(\theta_k - \frac{1}{1-s^e_i}\right)^2 z_j z_k^2 - \theta_k z_j z_k\left(\frac{dz_k}{ds^e_k}-\frac{dz_j}{ds^e_j}\right)\right]}{(1-b)m^*_i\left(\sum_{k=1}^n z_k\right)^3}\\
&= \frac{z_i\sum_{k=1}^n\sum_{j=k+1}^n z_j z_k\left[z_k\left(\theta_k - \frac{1}{1-s^e_i}\right)^2 + z_j\left(\theta_j - \frac{1}{1-s^e_i}\right)^2 + (\theta_j-\theta_k)\left(\frac{dz_k}{ds^e_k}-\frac{dz_j}{ds^e_j}\right)\right]}{(1-b)m^*_i\left(\sum_{k=1}^n z_k\right)^3}
\end{aligned}$$

where the inequality follows from discarding the partial derivatives of θ_k with respect to s^e_i, equal to $(1-s^e_i)^{-2}\geq 0$, and the corresponding terms $\frac{ds^e_i}{dm^*_i}$, which are positive by (A-7).

The weighted sum of squares between brackets is minimized for $\frac{1}{1-s^e_i} = \frac{z_k\theta_k + z_j\theta_j}{z_k+z_j}$ and hence the term in brackets is bounded from below by $(\theta_j-\theta_k)\left((\theta_j-\theta_k)\frac{z_k z_j}{z_k+z_j} + \frac{dz_k}{ds^e_k} - \frac{dz_j}{ds^e_j}\right)$. We have:

$$(\theta_j-\theta_k)\frac{z_k z_j}{z_k+z_j} + \frac{dz_k}{ds^e_k} - \frac{dz_j}{ds^e_j} = \frac{s^e_k s^e_j(1-bs^e_k)(1-bs^e_j)(s^e_k - s^e_j)}{G(s^e_j, s^e_k)}$$

where $G(s^e_j, s^e_k)$ is a cubic function of s^e_j and s^e_k. The numerator is nonnegative for any s^e_j, s^e_k, such that $0\leq s^e_j \leq s^e_k \leq 1$ and $s^e_j + s^e_k \leq 1$. Moreover, one can check that $G(s^e_j, s^e_j)\geq 0$, $G(s^e_j, 1-s^e_j)\geq 0$, and $dG(s^e_j, 1-s^e_j)/ds^e_k \leq 0$, which, together with the fact that the cubic coefficient of s^e_k is nonnegative, proves that $G(s^e_j, s^e_k)$ is nonnegative when $s^e_k \in [s^e_j, 1-s^e_j]$, and $s^e_j\in[0, 0.5]$.

Thus,

$$\frac{s^e_i}{bz_i}\frac{1-bs^e_i}{1-s^e_i} + \sum_{k=1}^n\left(\frac{1}{1-s^e_i}-\frac{1}{1-s^e_k}\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right)$$

is increasing and Π^e_{SC} is quasi-convex in m^*_i.

In fact, Π^e_{SC} may be increasing or decreasing in m^*_i. Suppose that $m^*_i \geq m^*_k$ for all k; then, $s^e_i \geq s^e_k$ for all k (from Proposition 1). In that case,

$$\sum_{k=1}^n \left(\frac{1}{1-s^e_k}\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right) \leq \frac{1}{1-s^e_i},$$

and thus Π^e_{SC} is increasing in m^*_i. Suppose on the other hand that $m^*_i = 0$; then $s^e_i = 0$ and $\frac{s^e_i}{z_i} = 1$, so that

$$\sum_{k=1}^n \left(\frac{1}{1-s^e_k}\right)\left(\frac{z_k}{\sum_{j=1}^n z_j}\right) \geq \frac{1}{n-1}\sum_{k\neq i}\left(\frac{1}{1-s^e_k}\right) \geq 1 + \frac{1}{n-2}.$$

In addition, suppose that $n = 2$. In that case, $1 + \frac{1}{n-2} \geq 1 + \frac{1}{b}$, and hence Π^e_{SC} is decreasing.

Since $m^*_i = a_i(r_i - c_i)$ and $m_i = a_i(r_i - w_i)$, we have the following:

(a) Equation (A-6) implies that w^e_i is increasing with c_i; $w_i = r_i - \frac{m_i(r_i - c_i)}{m^*_i} = r_i - \frac{b(1-s_i)(r_i-c_i)}{1-bs_i}$ and Equation (A-7) imply that w^e_i is increasing with r_i and a_i.

(b) Equation (A-6) implies that for $j \neq i$, w^e_j is increasing with c_i and decreasing with r_i and a_i.

(c) Equation (A-7) implies that s^e_i is decreasing with c_i and increasing with r_i and a_i.

(d) Equation (A-7) implies that, for $j \neq i$, s^e_j is increasing with c_i and decreasing with r_i and a_i.

(e) Equation (A-9) implies that Π^e_{Si} is decreasing with c_i and increasing with r_i and a_i.

(f) Equation (A-10) implies that, for $j \neq i$, Π^e_{Sj} is increasing with c_i and decreasing with r_i and a_i.

(g) Equation (A-11) implies that Π^e_R is decreasing with c_i and increasing with r_i and a_i.

(h) Since Π^e_{SC} is quasi-convex in m^*_i, it is quasi-convex in c_i, r_i and a_i.

■

Proof of Proposition 3

Proof. Similarly to the previous proof, differentiating implicitly Equation (11) yields

$$\frac{1}{m^e_i}\frac{dm^e_i}{db} = \frac{1}{b(1-bs^e_i)} - \frac{(1-b)}{(1-s^e_i)(1-bs^e_i)}\left(\frac{ds^e_i}{db}\right). \tag{A-13}$$

We have from Equation (3) that

$$\begin{aligned}\frac{1}{s_i}\frac{\partial s_i}{\partial b} &= \frac{\log(m_i)}{(1-b)^2} - s_i\left[\sum_{k=1}^n \frac{\log(m_k)}{(1-b)^2}\left(\frac{m_k}{m_i}\right)^{\frac{1}{1-b}}\right] = \frac{\sum_{k=1}^n \log(m_i)s_k}{(1-b)^2} - s_i\left[\sum_{k=1}^n \frac{\log(m_k)}{(1-b)^2}\frac{s_k}{s_i}\right]\\ &= \sum_{k=1}^n \frac{s_k\log\left(\frac{m_i}{m_k}\right)}{(1-b)^2} = \sum_{k=1}^n \frac{s_k\log\left(\frac{s_i}{s_k}\right)}{1-b} = \frac{\log(s_i)}{1-b} - \sum_{k=1}^n \frac{s_k\log(s_k)}{1-b}\end{aligned}$$

and hence

$$\frac{1}{s^e_i}\frac{ds^e_i}{db} = \frac{\log(s^e_i)}{1-b} - \sum_{k=1}^n \frac{s^e_k\log(s^e_k)}{1-b} + \frac{1}{(1-b)m^e_i}\frac{dm^e_i}{db} - \sum_{k=1}^n \frac{s^e_k}{(1-b)m^e_k}\frac{dm^e_k}{db}.$$

Substituting Equation (A-13) above yields

$$\begin{aligned}\frac{1}{s_i^e}\frac{ds_i^e}{db} &= \frac{\log(s_i^e)}{1-b} - \sum_{k=1}^n \frac{s_k^e\log(s_k^e)}{1-b} + \frac{1}{b(1-b)(1-bs_i^e)} - \frac{1}{(1-s_i^e)(1-bs_i^e)}\frac{ds_i^e}{db} \\ &\quad - \sum_{k=1}^n \frac{s_k^e}{b(1-b)(1-bs_k^e)} - \sum_{k=1}^n \frac{s_k^e}{(1-s_k^e)(1-bs_k^e)}\frac{ds_k^e}{db} \\ &= \frac{\frac{\log(s_i^e)}{1-b} + \frac{1}{b(1-b)(1-bs_i^e)} - \sum_{k=1}^n \frac{s_k^e\log(s_k^e)}{1-b} - \sum_{k=1}^n \frac{s_k^e}{b(1-b)(1-bs_k^e)} - \sum_{k=1}^n \frac{s_k^e}{(1-s_k^e)(1-bs_k^e)}\frac{ds_k^e}{db}}{1 + \frac{s_i^e}{(1-s_i^e)(1-bs_i^e)}}\end{aligned}$$

and hence,

$$\begin{aligned}\frac{(1-b)s_i^e}{(1-s_i^e)(1-bs_i^e)}\frac{ds_i^e}{db} &= \left(\frac{\left(s_i^e\right)^2}{s_i^e + (1-s_i^e)(1-bs_i^e)}\right) \\ &\quad \times \left(\log(s_i^e) + \frac{1}{b(1-bs_i^e)} - \sum_{k=1}^n s_k^e\log(s_k^e) - \sum_{k=1}^n \frac{s_k^e}{b(1-bs_k^e)} - \sum_{k=1}^n \frac{(1-b)s_k^e}{(1-s_k^e)(1-bs_k^e)}\frac{ds_k^e}{db}\right)\end{aligned}$$

Letting $z_k = \dfrac{\left(s_k^e\right)^2}{s_k^e + (1-s_k^e)(1-bs_k^e)} + s_k^e$, summing these identities for $i = 1,...,n$ yields

$$\sum_{k=1}^n \frac{(1-b)s_k^e}{(1-s_k^e)(1-bs_k^e)}\frac{ds_k^e}{db} = \left(\begin{array}{c}\displaystyle\sum_{k=1}^n (z_k - s_k^e)\log(s_k^e) + \frac{z_k - s_k^e}{b(1-bs_k^e)} \\ \displaystyle -\left(\sum_{k=1}^n (z_k - s_k^e)\right)\left(\sum_{k=1}^n s_k^e\log(s_k^e) + \frac{s_k^e}{b(1-bs_k^e)}\right)\end{array}\right) \times \left(\frac{1}{1 + \sum_{k=1}^n (z_k - s_k^e)}\right).$$

Thus,

$$\frac{(1-b)s_i^e}{(z_i - s_i^e)(1-s_i^e)(1-bs_i^e)}\frac{ds_i^e}{db} = \log(s_i^e) + \frac{1}{b(1-bs_i^e)} - \frac{\sum_{k=1}^n z_k\left(\log(s_k^e) + \frac{1}{b(1-bs_k^e)}\right)}{\sum_{k=1}^n z_k},$$

which gives rise to

$$\frac{ds_i^e}{db} = \left(\frac{2s_i^e - z_i}{b(1-b)}\right)\left(b\log(s_i^e) + \frac{1}{1-bs_i^e} - \frac{\sum_{k=1}^n z_k\left(b\log(s_k^e) + \frac{1}{1-bs_k^e}\right)}{\sum_{k=1}^n z_k}\right).$$

Hence, $\dfrac{ds_i^e}{db} \geq 0$ if and only if

$$b\log(s_i^e) + \frac{1}{1-bs_i^e} - \frac{\sum_{k=1}^n z_k\left(b\log(s_k^e) + \frac{1}{1-bs_k^e}\right)}{\sum_{k=1}^n z_k} \geq 0.$$

This implies that if $s_i^e \geq s_j^e$ and $\dfrac{ds_j^e}{db} \geq 0$, then $\dfrac{ds_i^e}{db} \geq 0$. Thus, as we increase b, the items with highest s_i^e gain space, while the others lose it. In addition, if supplier 1 receives the largest space allocation s_1^e, $\dfrac{ds_1^e}{db} \geq 0$, and if supplier n receives the smallest space allocation, s_n^e, $\dfrac{ds_n^e}{db} \leq 0$.

Let $\theta_k = b\log(s_k^e) + \dfrac{1}{1-bs_k^e}$ and sort the suppliers so that $s_1^e \geq \ldots \geq s_n^e$. Thus, $\theta_1 \geq \ldots \geq \theta_n$. When $\dfrac{ds_i^e}{db} = 0$, we know that $\dfrac{ds_k^e}{db} \geq 0$ for $k < i$ and $\dfrac{ds_k^e}{db} \leq 0$ for $k > i$. ■

Proof of Theorem 2

Proof. Putting together the equilibrium equations of Theorem 1 with Equation (3) implies that in equilibrium, $m_i^* = K s_i^{1-b}\left(\frac{1-bs_i}{b-bs_i}\right)$, and $m_i = K s_i^{1-b}$, where K is the same for all i. Equation (10) yields that the Price of Anarchy is the maximum over all instances of

$$\frac{\left[\sum_{i=1}^{n} s_i\left(\frac{1-bs_i}{b-bs_i}\right)^{\frac{1}{1-b}}\right]^{1-b}\left[\sum_{i=1}^{n} s_i\right]^{b}}{\sum_{i=1}^{n} s_i\left(\frac{1-bs_i}{b-bs_i}\right)} = \frac{\left[\sum_{i=1}^{n} s_i\left(\frac{1-bs_i}{1-s_i}\right)^{\frac{1}{1-b}}\right]^{1-b}}{\sum_{i=1}^{n} s_i\left(\frac{1-bs_i}{1-s_i}\right)}.$$

■

Supporting Lemma

Lemma 1. *The Price of Anarchy is given by* $s_2 = \ldots = s_n = \frac{z}{n-1}$ *and* $s_1 = 1-z$*. Thus,*

$$PoA_n = \max_{0\le z\le \frac{n-1}{n}, 0\le b\le 1} \frac{\left[z\left(b+\frac{1-b}{1-\frac{z}{n-1}}\right)^{\frac{1}{1-b}} + (1-z)\left(b+\frac{1-b}{z}\right)^{\frac{1}{1-b}}\right]^{1-b}}{b+(1-b)\left(\frac{z}{1-\frac{z}{n-1}}+\frac{1}{z}-1\right)}.$$

Proof. We can express the objective function in (14) as follows:

$$\frac{\left[\sum_{i=1}^{n-1} s_i\left(b+\frac{1-b}{1-s_i}\right)^{\frac{1}{1-b}} + \left(1-\sum_{i=1}^{n-1} s_i\right)\left(b+\frac{1-b}{\sum_{i=1}^{n-1} s_i}\right)^{\frac{1}{1-b}}\right]^{1-b}}{b+(1-b)\left[\sum_{i=1}^{n-1} s_i\left(\frac{1}{1-s_i}\right)+\left(1-\sum_{i=1}^{n-1} s_i\right)\left(\frac{1}{\sum_{i=1}^{n-1} s_i}\right)\right]} \doteq \frac{A^{1-b}}{B}$$

where A corresponds to the term in brackets in the numerator of the objective function of (14) and B corresponds to the term in the denominator.

For $i \le n-1$, the first-order condition with respect to s_i (log-derivative) is equivalent to

$$\frac{f(s_i)-f(s_n)}{g(s_i)-g(s_n)} = \frac{A}{B},$$

where

$$f(s) = \left(b+\frac{1-b}{1-s}\right)^{\frac{1}{1-b}}\left(1+\frac{s}{(1-s)(1-bs)}\right), \text{ and } g(s) = \left(\frac{1}{1-s}\right)^2.$$

Suppose $(s_1,...,s_n)$ satisfies the first-order optimality conditions. Hence any s_i and s_j, $j \ne i$, satisfy

$$[f(s_n)-f(s_j)]\,g(s_i) - [g(s_n)-g(s_j)] + g(s_n)f(s_j) - f(s_n)g(s_j) = 0.$$

Fix s_j and s_n, and denote by $F(s_i)$ the left-hand side of the above equation. It is easy to see that $F(s_i)$ has only one stationary point. Hence, it has at most two roots. In fact, it has exactly two roots, namely s_j and s_n. As a result, every optimal solution $(s_1,...,s_n)$, in the interior of the domain, has k components equal to s

and $(n-k)$ components equal to $(1-ks)/(n-k)$, for some feasible s. Without loss of generality, we assume that $s \geq (1-ks)/(n-k)$, i.e., $s \geq 1/n$.

The problem of finding PoA_n, (14), can therefore be reformulated as follows:

$$PoA_n = \max_{\frac{1}{n} \leq s \leq 1, 0 \leq b \leq 1, k \in \{1,\ldots,n-1\}} \frac{\left[ks\left(b + \frac{1-b}{1-s}\right)^{\frac{1}{1-b}} + (1-ks)\left(b + \frac{1-b}{1-\frac{1-ks}{n-k}}\right)^{\frac{1}{1-b}} \right]^{1-b}}{\left[ks\left(b + \frac{1-b}{1-s}\right) + (1-ks)\left(b + \frac{1-b}{1-\frac{1-ks}{n-k}}\right) \right]}.$$

When $s \geq 1/n$, the term $(b + (1-b)/(1-(1-ks/(n-k))))$ is decreasing in k, as well as the multiplicative coefficient $(1-ks)$. Thus, PoA_n is maximized when $k=1$, i.e., when $s_2 = \ldots = s_n$. Substituting s by $1-z$ leads to the lemma statement. From this formulation, it is easy to see that PoA_n is in fact strictly increasing with n, confirming a posteriori that a non-interior solution, $(s_1, \ldots, s_n)$ with $s_i = 0$ for some i, will never be optimal. ■

Proof of Theorem 3

Proof. The profit function of supplier i is

$$(w_i - c_i) a_i \Big(r_i(w_i)\Big)^{-\mu} s_i^b = \left(\frac{\mu}{\mu-1}\right)^{-\mu} a_i (w_i - c_i) w_i^{-\mu} \frac{\left(a_i w_i^{-(\mu-1)}\right)^{\frac{b}{1-b}}}{\left[\sum_{j=1}^n \left(a_j w_j^{-(\mu-1)}\right)^{\frac{1}{1-b}}\right]^b}$$

This is a quasi-concave function of w_i, and the optimal wholesale price is such that (log-differentiation)

$$\frac{1}{w_i - c_i} - \frac{\mu}{w_i} - \frac{(\mu-1)b(1-s_i)}{w_i(1-b)} = 0,$$

which is equivalent to Equation (15).

One can restrict the strategy space to a compact space, which, together with the quasiconcavity of the profit function, yields existence of a pure-strategy Nash equilibrium. Also, it is easy to see that $\frac{dw_i^{b.r.}}{dw_j} \geq 0$ for $j \neq i$. An argument similar to the one of Theorem 1 yields uniqueness. ■

Proof of Proposition 6

Proof. When the retailer decides the selling prices, $m_i^* = a_i(r_i(c_i) - c_i) r_i(c_i)^{-\mu} = a_i c_i^{-(\mu-1)} (\mu-1)^{\mu-1}/\mu^\mu$, and $m_i = a_i(r_i(w_i) - w_i) r_i(w_i)^{-\mu} = a_i w_i^{-(\mu-1)} (\mu-1)^{\mu-1}/\mu^\mu$. Hence, $m_i^* = m_i \left(\frac{w_i}{c_i}\right)^{\mu-1}$. From Equation (3), $m_i = K s_i^{1-b}$ and hence $m_i^* = K s_i^{1-b} \left(\frac{w_i}{c_i}\right)^{\mu-1}$, where K is the same for all i. Furthermore, from Equation (15), we have $m_i^* = K s_i^{1-b} \left(1 + \frac{1-b}{(\mu-1)(1-bs_i)}\right)^{\mu-1}$.

From Equation (9), the profit of the integrated supply chain equals

$$\Pi_{SC}^* = K \left[\sum_{i=1}^n s_i \left(1 + \frac{1-b}{(\mu-1)(1-bs_i)}\right)^{\frac{\mu-1}{1-b}}\right]^{1-b},$$

and the profit of the decentralized supply chain equals

$$\Pi_{SC}=\sum_{i=1}^{n}s_i^b\left(m_i+(w_i-c_i)a_ir_i(w_i)^{-\mu}\right)=\sum_{i=1}^{n}s_i^bm_i\left(1+\frac{w_i-c_i}{r_i(w_i)-w_i}\right)=K\sum_{i=1}^{n}s_i\left(1+\frac{1-b}{\frac{1-b}{\mu-1}+1-bs_i}\right).$$

The Price of Anarchy is the maximum ratio Π_{SC}^*/Π_{SC} over all problem instances, i.e., over all feasible b, μ, and $s_1,...,s_n$ subject to $\sum_{i=1}^{n}s_i=1$:

$$\max\frac{\left[\sum_{i=1}^{n}s_i\left(1+\frac{1-b}{(\mu-1)(1-bs_i)}\right)^{\frac{\mu-1}{1-b}}\right]^{1-b}}{\sum_{i=1}^{n}s_i\left(1+\frac{1-b}{\frac{1-b}{\mu-1}+1-bs_i}\right)}.$$

The maximum is in fact reached in the limit $\mu\to\infty$, and, since $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$, where e is the exponential number, it is equal to

$$\max\frac{\left[\sum_{i=1}^{n}s_ie^{\frac{1}{1-bs_i}}\right]^{1-b}}{\sum_{i=1}^{n}s_i\left(1+\frac{1-b}{1-bs_i}\right)}.$$

Here, the maximum can be shown to be reached for $b=0$, and hence yields $PoA=e/2$. Note that this bound is independent of n. ■

Proof of Proposition 7

Proof. Similarly to Theorem 1, there exists a unique Nash equilibrium in the game, given by the following conditions

$$\frac{r_i-w_i}{r_i-c_i}=\frac{b-bs_i}{1+b-bs_i}.$$

Similarly to the proof of Theorem 2, $m_i^*=\frac{Ks_i}{b}\left(b+\frac{1}{1-s_i}\right)$ and $m_i=Ks_i$ yields

$$PoA_n^{\$/sqft}=\max_{s_1,\dots,s_n,b}\left\{\frac{\left[\sum_{i=1}^{n}s_i^{\frac{1}{1-b}}\left(b+\frac{1}{1-s_i}\right)^{\frac{1}{1-b}}\right]^{1-b}}{\sum_{i=1}^{n}s_i^{1+b}\left(b+\frac{1}{1-s_i}\right)}\right\},$$

such that $\sum_{i=1}^{n}s_i=1$, $s_1,...,s_n\geq 0$, and $0\leq b\leq 1$.

We can easily show that the Price of Anarchy is reached with $s_1=\ldots=s_{n-1}=\frac{z}{n-1}$ and $s_n=1-z$, hence

$$PoA_n^{\$/sqft}=\max_{0\leq z\leq\frac{n-1}{n},0\leq b\leq 1}\left\{\frac{\left[(n-1)^{\frac{-b}{1-b}}z^{\frac{1}{1-b}}\left(b+\frac{1}{1-\frac{z}{n-1}}\right)^{\frac{1}{1-b}}+(1-z)^{\frac{1}{1-b}}\left(b+\frac{1}{z}\right)^{\frac{1}{1-b}}\right]^{1-b}}{(n-1)^{-b}z^{1+b}\left(b+\frac{1}{1-\frac{z}{n-1}}\right)+(1-z)^{1+b}\left(b+\frac{1}{z}\right)}\right\}.$$

This is increasing with n. For $n=2$, we obtain a lower bound:

$$PoA_2^{\$/sqft} = \max_{0\le z\le \frac{1}{2}, 0\le b\le 1} \left\{ \frac{\left[z^{\frac{1}{1-b}} \left(b+\frac{1}{1-z}\right)^{\frac{1}{1-b}} + (1-z)^{\frac{1}{1-b}} \left(b+\frac{1}{z}\right)^{\frac{1}{1-b}} \right]^{1-b}}{z^{1+b}\left(b+\frac{1}{1-z}\right) + (1-z)^{1+b}\left(b+\frac{1}{z}\right)} \right\}.$$

$PoA_2^{\$/sqft}$ is reached (maximized) with $b=1$, and we find numerically that $PoA_2^{\$/sqft} \in [1.298, 1.299]$. ■

Proof of Proposition 8

Let us denote with a PTS superscript the equilibrium profits associated with pay-to-stay fee contracts.

We have that

$$\Pi_{Si}^{PTS} = (r_i - c_i)a_i\left(s_i^*\right)^b - f s_i^* = m_i^*\left(s_i^*\right)^b - b\left[\sum_{j=1}^n \left(m_j^*\right)^{\frac{1}{1-b}}\right]^{1-b} s_i^* = (1-b)m_i^*\left(s_i^*\right)^b,$$

while the equilibrium profit without pay-to-stay fees was $\Pi_{Si}^e = (w_i^e - c_i)a_i\left(s_i^e\right)^b$. Using the optimality condition (11), we can express

$$\Pi_{Si}^e = \left(m_i^* - m_i^e\right)\left(s_i^e\right)^b = m_i^*\left(\frac{1-b}{1-bs_i^e}\right)\left(s_i^e\right)^b.$$

Hence, $\Pi_{Si}^{PTS} \le \Pi_{Si}^e$ if and only if

$$\left(s_i^*\right)^b \le \frac{\left(s_i^e\right)^b}{1-bs_i^e}. \tag{A-14}$$

From Equation (11), $m_i^* = m_i^e\left(\frac{1-bs_i^e}{b-bs_i^e}\right)$. Moreover, from Equation (3), we can express m_i^e as $K\left(s_i^e\right)^{1-b}$, where K is the same for all i. Therefore,

$$\begin{aligned}
\sum_{j=1}^n \left(m_j^*\right)^{\frac{1}{1-b}} &= \sum_{j=1}^n \left(m_j^e\right)^{\frac{1}{1-b}} \left(\frac{1-bs_j^e}{b-bs_j^e}\right)^{\frac{1}{1-b}} \\
&= \left(\frac{K}{b}\right)^{\frac{1}{1-b}} \sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}} \\
&= \left(\sum_{j=1}^n \left(m_j^e\right)^{\frac{1}{1-b}}\right) b^{\frac{-1}{1-b}} \sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}}.
\end{aligned}$$

This identity, together with Equations (3) and (11), yields

$$\begin{aligned}
\left(s_i^*\right)^{1-b} &= \frac{m_i^*}{\left(\sum_{j=1}^n \left(m_j^*\right)^{\frac{1}{1-b}}\right)^{1-b}} \\
&= \left(\frac{1-bs_i^e}{b-bs_i^e}\right) \frac{m_i^e b \left(\sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}}\right)^{-(1-b)}}{\left(\sum_{j=1}^n \left(m_j^e\right)^{\frac{1}{1-b}}\right)^{1-b}} \\
&= \left(\frac{1-bs_i^e}{1-s_i^e}\right)\left(s_i^e\right)^{1-b}\left(\sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}}\right)^{-(1-b)}.
\end{aligned}$$

For a given s_i^e, $\sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}}$ is minimized when $s_j^e=(1-s_i^e)/(n-1)$, for $j\neq i$, and when $n\to\infty$. Thus,

$$\sum_{j=1}^n s_j^e \left(\frac{1-bs_j^e}{1-s_j^e}\right)^{\frac{1}{1-b}} \geq s_i^e \left(\frac{1-bs_i^e}{1-s_i^e}\right)^{\frac{1}{1-b}} + (1-s_i^e).$$

Accordingly,

$$\left(s_i^*\right)^b \leq \left(\frac{1-bs_i^e}{1-s_i^e}\right)^{\frac{b}{1-b}} \left(s_i^e\right)^b \left(s_i^e \left(\frac{1-bs_i^e}{1-s_i^e}\right)^{\frac{1}{1-b}} + (1-s_i^e)\right)^{-b}.$$

One can check that $(1-bs_i^e)\left(s_i^e+(1-s_i^e)\left(\frac{1-s_i^e}{1-bs_i^e}\right)^{\frac{1}{1-b}}\right)^{-b} \leq 1$ for any s_i^e and b between 0 and 1, thereby proving inequality (A-14) and thus that $\Pi_i^{PTS}\leq\Pi_i^e$ for all i. Because the profits of all suppliers decrease, and the supply chain total profit increases, we must have that $\Pi_R^{PTS}\geq\Pi_R^e$. ■

Proof of Proposition 9

Since $w_1^{CM}=c_1$ and $w_i^{b.r}(w_{-i})$ is increasing in w_{-i} by (A-1), then in the category management equilibrium, $w_i^{CM}\leq w_i^e$. In addition, from Equation (11), we have that $s_i^{CM}\leq s_i^e$ for $i=2,...,n$, and therefore $s_1^{CM}\geq s_1^e$. This implies that $\Pi_{Si}^{CM}\leq\Pi_{Si}^e$ for $i=2,...,n$.

Since $\Pi_{S1}^{CM}+\Pi_R^{CM}=\max_{s_i}\left\{(r_1-c_1)a_1s_1^b+\sum_{i=2}^n(r_i-w_i^{CM})a_is_i^b\right\}$, $w_i^{CM}\leq w_i^e$ and $\Pi_{S1}^e+\Pi_R^e\leq \max_{s_i}\left\{(r_1-c_1)a_1s_1^b+\sum_{i=2}^n(r_i-w_i^e)a_is_i^b\right\}$, then clearly $\Pi_{S1}^{CM}+\Pi_R^{CM}\geq\Pi_{S1}^e+\Pi_R^e$. ■

Proof of Proposition 10

Proof. Let, without loss of generality, $m_1^*=1$. From (3), we have that

$$s_2^{CM}=\frac{\left(m_2^{CM}\right)^{\frac{1}{1-b}}}{1+\left(m_2^{CM}\right)^{\frac{1}{1-b}}},$$

hence $m_2^{CM}=\left(\frac{s_2^{CM}}{1-s_2^{CM}}\right)^{1-b}$. On the other hand, Equation (11) yields $m_2^{CM}=m_2^*\frac{b(1-s_2^{CM})}{1-bs_2^{CM}}$. The equilibrium space allocation thus satisfies $\left(\frac{s_2^{CM}}{1-s_2^{CM}}\right)^{1-b}=m_2^*\frac{b(1-s_2^{CM})}{1-bs_2^{CM}}$.

The PoA is equal to the maximum profit ratio over all problem instances, that is the maximum of

$$\frac{\Pi_{SC}^{CM}}{\Pi_{SC}^*}=\frac{\left(1-s_2^{CM}\right)^b+m_2^*\left(s_2^{CM}\right)^b}{\left(1-s_2^*\right)^b+m_2^*\left(s_2^*\right)^b}=\frac{\left(1-s_2^{CM}\right)^b+m_2^*\left(s_2^{CM}\right)^b}{\left(1+\left(m_2^*\right)^{\frac{1}{1-b}}\right)^{1-b}}$$

over all problem instances. Expressing m_2^* as a function of s_2^{CM} simplifies the problem to a two-variable optimization problem. Solving the resulting maximization problem over all feasible values of s_2 and b (between 0 and 1) yields that $PoA_2^{CM}\in[1.079,1.080]$. ■

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