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THE VALUE OF TAX SHIELDS DEPENDS  
ONLY ON THE NET INCREASES OF DEBT

The value of tax shields, the risk of the increases of debt and the risk of the  
increases of assets

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## **THE VALUE OF TAX SHIELDS DEPENDS ONLY ON THE NET INCREASES OF DEBT**

**The value of tax shields, the risk of the increases of debt and the risk of the  
increases of assets**

### **Abstract**

The value of tax shields depends only on the nature of the stochastic process of the net increases of debt. The value of tax shields in a world with no leverage cost is the tax rate times the current debt plus the present value of the net increases of debt. By applying this formula to specific situations, we show that Modigliani-Miller (1963) should be used when the company has a preset amount of debt; Fernández (2004), when the company maintains a fixed book-value leverage ratio; and Miles-Ezzell (1980), when the company maintains a fixed market-value leverage ratio.

*JEL classification:* G12; G31; G32

**Keywords:** Value of tax shields, present value of the net increases of debt, required return to equity

## THE VALUE OF TAX SHIELDS WITH A FIXED BOOK-VALUE LEVERAGE RATIO

THE VALUE OF TAX SHIELDS, THE RISK OF THE INCREASES OF DEBT  
AND THE RISK OF THE INCREASES OF ASSETS

There is no consensus in the existing literature regarding the correct way to compute the value of tax shields. Most authors think of calculating the value of the tax shield in terms of the appropriate present value of the tax savings due to interest payments on debt, but Modigliani-Miller (1963) propose to discount the tax savings at the risk-free rate<sup>1</sup>, whereas Harris and Pringle (1985) propose discounting these tax savings at the cost of capital for the unlevered firm. Miles and Ezzel (1985) propose discounting these tax savings the first year at the cost of debt and the following years at the cost of capital for the unlevered firm. Reflecting this lack of consensus, Copeland et al. (2000, p. 482) claim that “the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct.”

We show that the value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. More specifically, we prove that the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the present value of the net increases of debt.

By applying this formula to specific situations, we show that the Modigliani-Miller (1963) formula should be used when the company has a preset amount of debt; Fernández (2004), when the company expects the increases of debt to be as risky as the free cash flows (for example, if the company wants to maintain a fixed book-value leverage ratio); and Miles-Ezzell (1980), only if debt will be always a multiple of the equity market value  $D_t = L \cdot E_t$ . We will argue that although  $D_t = L \cdot E_t$  provides a computationally elegant solution, it is not a realistic one. What is more, we have not seen any company that follows this financing policy.

It makes much more sense to characterize the debt policy of a company with expected constant leverage ratio as a fixed book-value leverage ratio rather than as a fixed market-value leverage ratio because

1. the debt does not depend on the movements of the stock market,
2. it is easier to follow for unlisted companies, and
3. managers should prefer it because the value of tax shields is higher.

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<sup>1</sup> Myers (1974) propose to discount it at the cost of debt ( $K_d$ ).

Although Cooper and Nyborg (2006) disagree, this paper shows that Fernández's (2004) formula (28) ( $VTS = PV[Ku; D \cdot T \cdot Ku]$ ) is valid, but only under the assumption that the increases of debt are as risky as the free cash flows. The increases of debt are as risky as the free cash flows if the company maintains a fixed book-value leverage ratio.

## 1. General expression of the value of tax shields

The present value of debt (D) plus that of the equity (E) of the levered company is equal to the value of the unlevered company ( $V_u$ ) plus the value of tax shields due to interest payments (VTS):

$$E + D = V_u + VTS. \quad (1)$$

In the literature, the value of tax shields defines the increase in the company's value as a result of the tax saving obtained by the payment of interest. If leverage costs do not exist, then Eq. (1) could be stated as follows:

$$V_u + G_u = E + D + GL \quad (2)$$

where  $G_u$  is the present value of the taxes paid by the unlevered company and  $GL$  is the present value of the taxes paid by the levered company. Eq. (2) means that the total value of the unlevered company (left-hand side of the equation) is equal to the total value of the levered company (right-hand side of the equation). Total value is the enterprise value (often called the value of the firm) plus the present value of taxes. Eq. (2) assumes that expected free cash flows are independent of leverage<sup>2</sup>.

From (1) and (2), it is clear that VTS is

$$VTS = G_u - GL \quad (4)$$

Note that the value of tax shields is the difference between the PVs of two flows with different risk: the PV of the taxes paid by the unlevered company ( $G_u$ ) and the PV of the taxes paid by the levered company ( $GL$ ).

It is quite easy to prove that the relationship between the profit after tax of the levered company ( $PAT_{Lt}$ ) and the equity cash flow (ECF) is:

$$ECF_t = PAT_{Lt} - \Delta A_t + \Delta D_t \quad (5)$$

Notation being,  $\Delta A_t$  = Increase of net assets in period t (Increase of Working Capital Requirements plus Increase of Net Fixed Assets);  $\Delta D_t = D_t - D_{t-1}$  = Increase of Debt in period t.

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<sup>2</sup> When leverage costs do exist, the total value of the levered company is lower than the total value of the unlevered company. A world with leverage cost is characterized by the following relation:

$$V_u + G_u = E + D + GL + \text{Leverage Cost} > E + D + GL \quad (3)$$

Leverage cost is the reduction in the company's value due to the use of debt.

Similarly, the relationship between the profit after tax of the unlevered company (PAT<sub>u</sub>) and the free cash flow (FCF) is:

$$FCF_t = PAT_{u_t} - \Delta A_t \quad (6)$$

The taxes paid every year by the unlevered company (Taxes<sub>U</sub>) are

$$Taxes_{U_t} = [T/(1-T)] PAT_{u_t} = [T/(1-T)] (FCF_t + \Delta A_t) \quad (7)$$

For the levered company, taking into consideration Eq. (5), the taxes paid each year (Taxes<sub>L</sub>) are:

$$Taxes_{L_t} = [T/(1-T)] (ECF_t + \Delta A_t - \Delta D_t) \quad (8)$$

PV<sub>0</sub>[·] is the present value operator. The present values at t=0 of equations (7) and (8) are:

$$G_{u_0} = [T/(1-T)] (V_{u_0} + PV_0[\Delta A_t]) \quad (9)$$

$$G_{L_0} = [T/(1-T)] (E_0 + PV_0[\Delta A_t] - PV_0[\Delta D_t]) \quad (10)$$

The increase in the company's value due to the use of debt is the difference between G<sub>u</sub> (9) and G<sub>L</sub> (10), which are the present values of two cash flows with different risks:

$$VTS_0 = G_{u_0} - G_{L_0} = [T/(1-T)] (V_{u_0} - E_0 + PV_0[\Delta D_t]) \quad (11)$$

As, according to equation (1),  $V_{u_0} - E_0 = D_0 - VTS_0$ , then

$VTS_0 = [T/(1-T)] (D_0 - VTS_0 + PV_0[\Delta D_t])$ . And the value of tax shields is:

$$VTS_0 = T \cdot D_0 + T \cdot PV_0[\Delta D_t] \quad (12)$$

Equation (12) is valid for perpetuities and for companies with any pattern of growth<sup>3</sup>. More importantly, this equation shows that the value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. The problem of equation (12) is how to calculate PV<sub>0</sub>[ΔD<sub>t</sub>], which requires to know the appropriate discount rate to apply to the expected increase of debt.<sup>4</sup>

<sup>3</sup> Equation (12) may also be deduced in a very straightforward way. The value of the debt today is the present value of the interest minus the present value of the increases of debt:  $D_0 = PV_0[int_t] - PV_0[\Delta D_t]$ . As the value of tax shields is the present value of the interest times the tax rate,  $VTS = T \cdot PV_0[int_t] = T \cdot D_0 + T \cdot PV_0[\Delta D_t]$ .

<sup>4</sup> If the nominal value of debt (N) is not equal to the value of debt (D), because the interest rate (r) is different from the required return to debt flows (Kd), equation (12) is:  $VTS_0 = T \cdot D_0 + T \cdot PV_0[\Delta N_t]$ . The relationship between D and N is:  $D_0 = PV_0[N_t \cdot r_t] - PV_0[\Delta N_t]$ . From this equation we may also arrive at equation (12) because  $PV_0[N_t \cdot r_t] = D_0 + PV_0[\Delta N_t]$ . Multiplying both sides by T, we get equation (12):  $VTS = PV_0[N_t \cdot r_t \cdot T] = T \cdot D_0 + T \cdot PV_0[\Delta N_t]$ .

We may not know what are the correct values of  $G_U$  and  $G_L$ , but we know the value of the difference, provided we can value  $PV_0[\Delta D_t]$ , the present value of the net debt increases<sup>5</sup>.

## 2. VTS in specific situations

To develop a better understanding of the result in (12), we apply it in specific situations and show how this formula is consistent with previous formulae under restrictive scenarios.

### 2.1. Debt of one-year maturity but perpetually rolled over

As in the previous case,  $E_0[D_t] = D_0$ , but the debt is expected to be rolled over every year. The appropriate discount rate for the cash flows due to the existing debt is  $K_d$ <sup>6</sup>. Define  $K_{ND}$  as the appropriate discount rate for the new debt (the whole amount) that must be obtained every year, then:

Present value of obtaining the new debt every year<sup>7</sup> =  $D_0 / K_{ND}$

Present value of the principal repayments at the end of every year<sup>8</sup> =  $D_0 (1 + K_{ND}) / [(1 + K_d) K_{ND}]$

$PV_0[\Delta D_t]$  is the difference of the last two expressions. Then:

$$PV_0[\Delta D_t] = -D_0 (K_{ND} - K_d) / [(1 + K_d) K_{ND}] \quad (13)$$

In a constant perpetuity ( $E_0[FCF_t] = FCF_0$ ), it may be reasonable that, if we do not expect credit rationing,  $K_{ND} = K_d$ , which means that the risk associated with the repayment of the current debt and interest ( $K_d$ ) is equivalent to the risk associated with obtaining an equivalent amount of debt at the same time ( $K_{ND}$ ).

### 2.2. Debt is proportional to the Equity market value

This is the assumption made by Miles and Ezzell (1980) and Arzac and Glisten (2005), who show that if  $D_t = L \cdot E_t$ , then the value of tax shields for perpetuities growing at a constant rate  $g$  is:

<sup>5</sup> Fernández (2004) neglected to include in Equations (5) to (14) terms with expected value equal to zero. And he wrongly considered as being zero the present value of a variable with expected value equal to zero. Due to these errors, Equations (5) to (17), Tables 3 and 4, and Figure 1 of Fernández (2004) are correct only if  $PV_0[\Delta A_t] = PV_0[\Delta D_t] = 0$ .

<sup>6</sup> We use  $K_d$  so as not to complicate the notation. It should be  $K_{d,t}$ , a different rate following the yield curve. Using  $K_d$  we may also think of a flat yield curve.

<sup>7</sup> Present value of obtaining the new debt every year =  $D / (1 + K_{ND}) + D / (1 + K_{ND})^2 + D / (1 + K_{ND})^3 + \dots$  because  $D = E\{D_t\}$ , where  $D_t$  is the new debt obtained at the end of year  $t$  (beginning of  $t+1$ ).

<sup>8</sup> The present value of the principal repayment at the end of year 1 is  $D / (1 + K_d)$   
The present value of the principal repayment at the end of year 2 is  $D / [(1 + K_d)(1 + K_{ND})]$   
The present value of the principal repayment at the end of year  $t$  is  $D / [(1 + K_d)(1 + K_{ND})^{t-1}]$   
Because  $D = E\{D_t\}$ , where  $D_t$  is the debt repayment at the end of year  $t$ .

$$VTS_0 = \frac{D_0 Kd T (1 + Ku)}{(Ku - g) (1 + Kd)} \quad (14)$$

Substituting (14) in (12), we get:

$$PV_0[\Delta D_t] = D_0 \frac{(Kd - Ku) + g(1 + Kd)}{(Ku - g)(1 + Kd)} \quad (15)$$

For the no growth case ( $g = 0$ ), equation (15) is:

$$PV_0[\Delta D_t] = D (Kd - Ku) / [Ku(1 + Kd)] < 0.$$

Comparing this expression with equation (13), it is clear that Miles and Ezzell imply that  $K_{ND} = Ku$ . The Miles-Ezzell setup works as if the company pays all the debt ( $D_{t-1}$ ) at the end of every period  $t$  and simultaneously raises all new debt  $D_t$ . The risk of raising the new debt is similar to the risk of the free cash flow and, hence, the appropriate discount rate for the expected value of the new debt is  $Ku$ .

However, to assume  $D_t = L \cdot E_t$  is not a good description of the debt policy of any company because if a company has only two possible states of nature in the following period, it is clear that under the worst state (low share price) the leveraged company will have to raise new equity and repay debt, and this is not the moment companies prefer to raise equity. Under the good state, the company will have to take a lot of debt and pay big dividends.

$D_t = L \cdot E_t$  provides a computationally elegant solution (as shown in Arzac-Glosten, 2005), but unfortunately not a realistic one. Furthermore, we have not seen any company that follows this financing policy.

In Appendix 1 we prove that if  $D_t = L \cdot E_t$ , then the appropriate discount rate for the expected taxes is equal for the levered and the unlevered firm ( $Ku$ ) for  $t > 1$ .

### ***2.3. Debt is proportional to the Equity book value***

It makes more sense to characterize the debt policy of a growing company with expected constant leverage ratio as a fixed book-value leverage ratio rather than as a fixed market-value leverage ratio because:

1. the debt does not depend on the movements of the stock market,
2. it is easier to follow for unlisted companies, and
3. managers should prefer it because the value of tax shields is higher.

If  $D_t = K \cdot E_{bv_t}$ , where  $E_{bv}$  is the book value of equity, then  $\Delta D_t = K \cdot \Delta E_{bv_t}$ . The increase in the book value of equity is equal to the profit after tax (PAT) minus the equity cash flow. According to equation (5),

$$\Delta E_{bv_t} = PAT_{L_t} - ECF_t = \Delta A_t - \Delta D_t = \Delta D_t / K \quad (16)$$



In this situation, the increase of debt is proportional to the increases of net assets, and the risk of the increases of debt is equal to the risk of the increases of assets:

$$\Delta D_t = \Delta A_t / (1+1/K) \quad (17)$$

If  $\alpha$  is the appropriate discount rate for the expected increases of assets, then the present value of the increases of debt of a constant growing perpetuity is

$$PV_0[\Delta D_t] = \frac{gD_0}{(\alpha - g)} \quad (18)$$

And, substituting (18) in (12), the VTS is:

$$VTS_0 = \frac{D_0 \alpha T}{(\alpha - g)} \quad (19)$$

#### ***2.4. Debt is proportional to the Equity book value. Debt increases are as risky as the free cash flows***

If we also assume that the risk of the increases of net assets is equal to the risk of the free cash flow, then the increases of the debt are as risky as the free cash flows ( $\alpha = K_u$ ). In this situation, the correct discount rate for the expected increases of debt is  $K_u$ , the required return to the unlevered company. In the case of a constant growing perpetuity,  $PV_0[\Delta D_t] = g \cdot D_0 / (K_u - g)$ , and the VTS is Equation (28) in Fernández (2004):

$$VTS_0 = D_0 \cdot K_u \cdot T / (K_u - g) \quad (20)$$

#### ***2.5. The company has a preset amount of debt***

In this case, the appropriate discount rate for the  $\Delta D_t$  (known with certainty today) is  $R_F$ , the risk-free rate. In this situation, Modigliani-Miller (1963) applies and the VTS for a growing perpetuity, according to equation (12), is:

$$VTS_0 = D_0 \cdot T + T \cdot g \cdot D_0 / (R_F - g) = T \cdot D_0 \cdot R_F / (R_F - g) \quad (21)$$

Note that, in the case a growing perpetuity, Modigliani-Miller is just one case of section 2.3, in which  $\alpha = R_F$ .

### **3. Value of net debt increases implied by other authors**

Table I summarizes the implications of several approaches for calculating the value of tax shields. From equation (12), the present value of the increases of debt is:

$$PV_0[\Delta D_t] = (VTS_0 - T \cdot D_0) / T$$

Applying this equation to the theories mentioned, we may construct the predictions that each of these theories have for  $PV_0[\Delta D_t]$ .

As we have already argued, Modigliani-Miller (1963) should be used when the company has a preset amount of debt; Fernández (2004), when we expect the increases of debt to be as risky as the free cash flow (for example, if the company wants to maintain a fixed book-value leverage ratio); and Miles-Ezzell (1980), only if debt will be a multiple of the equity market value  $D_t = L \cdot E_t$ . If the company maintains a fixed book-value leverage ratio and the risk of the increases of assets is different than the risk of the free cash flow, then the formulas of section 2.3 (and Appendix A2) should be applied.

Fieten et al. (2005) argue that the Modigliani-Miller formula may be applied to all situations. We have shown that it is valid only when the company has a preset amount of debt.

Cooper and Nyborg (2006) affirm that equation (18) violates value-additivity. It does not because Equation (1) holds. They use only the cost of debt ( $R_F$ ) or the cost of the unlevered equity ( $K_u$ ) to discount the expected value of tax shields. We have seen that there are also other debt policies, such as when the firm wants to maintain a fixed book-value leverage ratio.

#### 4. A numerical example and a closer look at the discount rates

Appendixes 1, 2 and 3 derive additional formulae for the three theories discussed in this paper: Miles-Ezzell, Fernández and Modigliani-Miller, applied to growing perpetuities. Table II is a summary of the main formulae. Table III contains the main valuation results for a constant growing company. It is interesting to note that according to Miles-Ezzell, the present value of the increases of debt is negative. It is negative if  $g < (K_u - R_F)/(1 + R_F)$ .

First, we derive the expression for the value of tax shields. Table IV contains the value of the tax shields (VTS) according to the different theories as a function of  $g$  and  $\alpha$ . The results change dramatically when  $g$  increases. It may be seen that Modigliani-Miller is equivalent to a constant book-value leverage ratio ( $D_t = L \cdot E_{bv_t}$ ), when  $\alpha = R_F = 5\%$ . Fernández (2004) is equivalent to  $D_t = L \cdot E_{bv_t}$  when  $\alpha = K_u = 9\%$ .

Second, we derive the appropriate discount rates for the increases of debt. It is interesting to note that while two theories assume a constant rate (Modigliani-Miller assume  $R_F$  and Fernández assumes  $K_u$ ), Miles-Ezzell assume one rate for  $t = 1$  and  $K_u$  for  $t > 1$ . The appropriate discount rate for the increase of debt at  $t = 1$  is, according to Miles-Ezzell, equation (A1.2):

$$1 + K_{\Delta D1} = \frac{g(1 + K_u)(1 + R_F)}{g(1 + R_F) + R_F - K_u}$$

In our example,  $K_{\Delta D1} = -220.5\%$ .

Table V contains the present value of the increases of debt in different periods and the sum of all of them. According to Miles-Ezzell, the present value of the increases of debt in every period is negative.

We also prove that although the equity value of a growing perpetuity can be computed by discounting the expected value of the equity cash flow with a single rate  $K_e$ , the appropriate discount rates for the expected values of the equity cash flows are not constant. Table VI presents the appropriate discount rates for the expected values of the equity cash flows of our example. According to Miles-Ezzell,  $K_{e_t}$  is 58.3% for  $t = 1$  and 9% for the rest of the periods.

We also derive the appropriate discount rates for the expected values of the taxes. If we assume that the appropriate discount rate for the increases of assets is  $K_u$ , then the appropriate discount rate for the expected value of the taxes of the unlevered company is also  $K_u$ . But the appropriate discount rate for the expected value of the taxes of the levered company ( $K_{e_{TAXL}}$ ) is different according to the three theories. Table VII presents the appropriate discount rates for the expected values of the taxes in the initial periods for our example. According to Miles-Ezzell,  $K_{e_{TAXL_t}}$  is 9.79% for  $t = 1$  and 9% for the rest of the periods. According to the other theories,  $K_{e_{TAXL_t}}$  grows with  $t$ .

According to Modigliani-Miller and according to Fernández, the taxes of the levered company are riskier than the taxes of the unlevered company. However, according to Miles-Ezzell, both taxes are equally risky for  $t > 1$ .<sup>9</sup>

## 6. Is $K_u$ independent of growth?

Up to now we have assumed that  $K_u$  is constant, independent of growth. From equation (6) we know that  $FCF_t = PATu_t - \Delta A_t$ .

If we consider that the risk of the unlevered profit after tax ( $PATu$ ) is independent of growth, and that  $K_{PATu}$  is the required return to the expected  $PATu$ , the present value of equation (6) is:

$$V_{u_0} = \frac{(1+g)FCF_0}{(K_u - g)} = \frac{(1+g)PATu_0}{(K_{PATu} - g)} - \frac{gA_0}{(\alpha - g)}$$

$$K_u = g + \frac{(1+g)FCF_0}{\frac{(1+g)PATu_0}{(K_{PATu} - g)} - \frac{gA_0}{(\alpha - g)}}$$

Table VIII contains the required return to the free cash flows ( $K_u$ ) as a function of  $\alpha$  (required return to the increase of assets) and  $g$  (expected growth). It may be seen that  $K_u$  is increasing in  $g$ <sup>10</sup> if  $\alpha < K_{PATu}$ , and decreasing in  $g$  if  $\alpha > K_{PATu}$ .

<sup>9</sup> If the risk of the increase of assets is smaller than the risk of the free cash flows, then Miles-Ezzell provides a surprising result: the taxes of the levered company are less risky than the taxes of the unlevered company.

<sup>10</sup> This result contradicts Cooper and Nyborg (2006), who maintain that “ $K_u$  is decreasing in  $g$ ”.

## 7. Conclusions

The value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. More specifically, the value of tax shields in a world with no leverage cost is the tax rate times the current debt, plus the tax rate times the present value of the net increases of debt. This expression is the difference between the present values of two different cash flows, each with its own risk: the present value of taxes for the unlevered company and the present value of taxes for the levered company. The critical parameter for calculating the value of tax shields is the present value of the net increases of debt. It may vary for different companies, but it may be calculated in specific circumstances.

For perpetual debt, the value of tax shields is equal to the tax rate times the value of debt. When the debt level is fixed, Modigliani-Miller (1963) applies, and the value of tax shields is the present value of the tax shields, discounted at the required return to debt. If the leverage ratio ( $D/E$ ) is fixed at market value, then Miles-Ezzell (1980) applies, with the caveats discussed. If the leverage ratio is fixed at book values and the increases of assets are as risky as the free cash flows (the increases of debt are as risky as the free cash flows), then Fernández (2004) applies. We have developed new formulas for the situation in which the leverage ratio is fixed at book values but the increases of assets have a different risk than the free cash flows.

**Table I**  
**Present value of the increases of debt implicit in the most popular formulae for calculating the value of tax shields.  $VTS_0 = D_0 \cdot T + T \cdot PV_0[\Delta D_t]$**   
**Perpetuities growing at a constant rate  $g$**

Authors	$VTS_0$	$PV_0[\Delta D_t]$
Miles-Ezzell (1980) Arzac-Glosten (2005)	$\frac{D_0 R_F T (1 + K_u)}{(K_u - g) (1 + R_F)}$	$\frac{g \cdot D_0}{K_u - g} - \frac{D_0 (K_u - R_F)}{(K_u - g)(1 + R_F)}$
Modigliani-Miller (1963)	$\frac{D_0 R_F T}{(R_F - g)}$	$\frac{g \cdot D_0}{R_F - g}$
Fernández (2004)	$\frac{D_0 K_u T}{(K_u - g)}$	$\frac{g \cdot D_0}{K_u - g}$
Constant book-value leverage	$\frac{D_0 \alpha T}{(\alpha - g)}$	$\frac{g \cdot D_0}{\alpha - g}$

$K_u$  = unlevered cost of equity

$T$  = corporate tax rate

$D_0$  = debt value today

$R_F$  = risk-free rate

$\alpha$  = required return to the increases of assets

**Table II**  
**Main formulas in the appendixes for growing perpetuities**

	Modigliani-Miller	Fernández (2004)	Miles-Ezzell
	$D_t$ fixed $\Delta D_t = K \cdot CF_d$	$D_t = K \cdot E_b v_t$ $\Delta D_t = K \cdot FCF_t$	$D_t = K \cdot E_t$ $\Delta D_t = K \cdot \Delta FCF_t$
$VTS_0$	$\frac{D_0 R_F T}{(R_F - g)}$	$\frac{D_0 K u T}{(K u - g)}$	$\frac{D_0 R_F T}{(K u - g)} \frac{(1 + K u)}{(1 + R_F)}$
$1 + K_{\Delta D1}$	$1 + R_F$	$1 + K u$	$\frac{g(1 + K u)(1 + R_F)}{g(1 + R_F) + R_F - K u}$
$K_{\Delta D2}$	$R_F$	$K u$	$K u$
$K_e$	$K u + \frac{D_0}{E_0} (K u - R_F) \left(1 - \frac{R_F T}{(R_F - g)}\right)$	$K u + \frac{D_0}{E_0} (K u - R_F) (1 - T)$	$K u + \frac{D_0}{E_0} (K u - R_F) \left[1 - \frac{R_F T}{1 + R_F}\right]$
$K_{TS1}$	$R_F$	$R_F$	$R_F$
$K_{TS2}$	$R_F$	$\frac{R_F(1 + K u) + g K u(1 + R_F)}{(1 + K u) + g(1 + R_F)}$	$K u$

	<b><math>1 + K_e</math></b>
Miles-Ezzell	$\frac{(K_e - g)(1 + K u)}{(K u - g)} \quad (A1.6)$
Modigliani-Miller	$\frac{(K_e - g)(1 + R_F)(1 + K u)}{(K u - g)(1 + R_F) + (K_e - K u)(1 + g)} \quad (A3.3)$
Fernández (2004)	$\frac{(K_e - g)(1 + R_F)(1 + K u)}{(K u - g)(1 + R_F) + (K_e - K u)} \quad (A2.6)$
$D_t = L \cdot E_b v_t$	$\frac{E_0 (K_e - g)}{V_{u0} \frac{(K u - g)}{(1 + K u)} + D_0 \left[ \frac{g}{(1 + \alpha)} - \frac{R_F(1 - T)}{(1 + R_F)} \right]} \quad (A2.5)$

**Table III**  
**Example. Valuation of a constant growing company**

$FCF_0 = 50$ ;  $A_0 = 1000$ ;  $D_0 = 375$ ;  
 $R_F = 5\%$ ;  $K_U = 9\%$ ;  $T = 40\%$ ;  $g = 2\%$ .

	<b>Modigliani-Miller</b>	<b>Fernández</b>	<b>Miles-Ezzell</b>
	<b><math>D_t</math> fixed</b> $\Delta D_t = K \cdot CF_{d,t}$	<b><math>D_t = K \cdot E_{b,t}</math></b> $\Delta D_t = K \cdot FCF_t$	<b><math>D_t = K \cdot E_t</math></b> $\Delta D_t = K \cdot \Delta FCF_t$
$D_0$	375	375	375
$V_{U_0}$	728.57	728.57	728.57
$VTS_0$	250.0	192.86	111.23
$E_0$	603.57	546.43	464.80
$PV_0[\Delta D_t]$	250.0	107.14	-96.94
$G_U = PV_0[TAX_{U,t}]$	676.19	676.19	676.19
$G_L = PV_0[TAX_{L,t}]$	426.19	483.33	564.96
$K_e$	9.83%	10.65%	12.17%

	<b>Modigliani-Miller</b>		<b>Fernández</b>		<b>Miles-Ezzell</b>	
	<b><math>D_t</math> fixed</b>		<b><math>D_t = K \cdot E_{b,t}</math></b>		<b><math>D_t = K \cdot E_t</math></b>	
	<b>t=1</b>	<b>t=2</b>	<b>t=1</b>	<b>t=2</b>	<b>t=1</b>	<b>t=2</b>
$K_{\Delta D_t}$	5%	5%	9%	9%	-220.5%	9%
$K_{D_t}$	5%	5%	5.076%	5.147%	9%	9%
$K_{e_t}$	9.331%	9.344%	9.998%	10.025%	58.295%	9%
$K_{E_t}$	9.867%	9.907%	10.702%	10.761%	9%	9%
$K_{V_{ut}}$	9%	9%	9%	9%	9%	9%
$K_{VTS_t}$	5%	5%	5.923%	5.956%	9%	9%
$K_{TS_t}$	5%	5%	5%	5.076%	5%	9%
$K_{TAX_{U,t}}$	9%	9%	9%	9%	9%	9%
$K_{TAX_{L,t}}$	9.787%	9.824%	9.787%	9.807%	9.787%	9%
<b><math>K_e</math> average</b>	9.828%		10.647%		12.166%	

**Table IV**  
**Present value of the tax shields (VTS) according to the different theories as a function of  $g$  (expected growth) and  $\alpha$  (required return to the increase of assets).**  
 **$D_0 = 375$ ;  $R_F = 5\%$ ;  $K_u = 9\%$ ;  $T = 40\%$**

	$g$					
	0%	1%	2%	3%	4%	5%
Miles-Ezzell	86.51	97.32	111.22	129.76	155.71	194.64
Modigliani-Miller	150.00	187.50	250.00	375.00	749.95	7142.86
Fernández (2004)	150.00	168.75	192.86	225.00	270.00	337.50
$D_t = L \cdot E_{bv_t}$ ; $\alpha=5\%$	150.00	187.50	250.00	375.00	749.95	7142.86
$D_t = L \cdot E_{bv_t}$ ; $\alpha=7\%$	150.00	175.00	210.00	262.50	350.00	525.00
$D_t = L \cdot E_{bv_t}$ ; $\alpha=9\%$	150.00	168.75	192.86	225.00	270.00	337.50
$D_t = L \cdot E_{bv_t}$ ; $\alpha=11\%$	150.00	165.00	183.33	206.25	235.71	275.00
$D_t = L \cdot E_{bv_t}$ ; $\alpha=15\%$	150.00	160.71	173.08	187.50	204.55	225.00

**Table V**  
**Present value of the increases of debt in different periods and the sum of all of them.**  
 **$D_0 = 375$ ;  $R_F = 5\%$ ;  $K_u = 9\%$ ;  $T = 40\%$ ;  $g = 2\%$**

$PV_0(\Delta D_t)$	t=1	t=2	t=3	t=4	t=5	t=10	t=20	t=30	t=40	t=50	Sum
Miles-Ezzell	-6.23	-5.83	-5.45	-5.10	-4.77	-3.43	-1.76	-0.91	-0.47	-0.24	-96.94
Modigliani-Miller	7.14	6.94	6.74	6.55	6.36	5.50	4.12	3.08	2.31	1.73	250.00
Fernández (2004)	6.88	6.44	6.03	5.64	5.28	3.79	1.95	1.00	0.52	0.27	107.14
$D_t = L \cdot E_{bv_t}$ ; $\alpha=5\%$	7.14	6.94	6.74	6.55	6.36	5.50	4.12	3.08	2.31	1.73	250.00
$D_t = L \cdot E_{bv_t}$ ; $\alpha=7\%$	7.01	6.68	6.37	6.07	5.79	4.56	2.82	1.75	1.08	0.67	150.00
$D_t = L \cdot E_{bv_t}$ ; $\alpha=9\%$	6.88	6.44	6.03	5.64	5.28	3.79	1.95	1.00	0.52	0.27	107.14
$D_t = L \cdot E_{bv_t}$ ; $\alpha=11\%$	6.76	6.21	5.71	5.24	4.82	3.16	1.36	0.58	0.25	0.11	83.33

**Table VI**  
**Appropriate discount rates for the expected values of the equity cash flows ( $Ke_t$ )**  
 **$FCF_0 = 50$ ;  $D_0 = 375$ ;  $R_F = 5\%$ ;  $K_u = 9\%$ ;  $T = 40\%$ ;  $g = 2\%$ .**

$Ke_t$	t=1	t=2	t=3	t=5	t=10	t=20	t=30	t=50
Miles-Ezzell	58.30%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
Modigliani-Miller	9.33%	9.34%	9.36%	9.39%	9.48%	9.74%	10.17%	12.65%
Fernández (2004)	10.00%	10.02%	10.05%	10.11%	10.28%	10.76%	11.55%	17.37%
$D_t = L \cdot E_{bv_t}$ ; $\alpha=5\%$	9.33%	9.34%	9.36%	9.39%	9.48%	9.74%	10.17%	12.65%
$D_t = L \cdot E_{bv_t}$ ; $\alpha=7\%$	9.67%	9.70%	9.72%	9.78%	9.95%	10.42%	11.22%	17.42%
$D_t = L \cdot E_{bv_t}$ ; $\alpha=9\%$	10.00%	10.02%	10.05%	10.11%	10.28%	10.76%	11.55%	17.37%
$D_t = L \cdot E_{bv_t}$ ; $\alpha=11\%$	10.32%	10.33%	10.35%	10.40%	10.52%	10.88%	11.50%	15.54%



**Table VII**

**Appropriate discount rate for the expected value of the taxes of the levered company.**

$\alpha = K_u = 9\%$ ;  $FCF_0 = 50$ ;  $D_0 = 375$ ;  $R_F = 5\%$ ;  $T = 40\%$ ;  $g = 2\%$ .

$K_{TAXL_t}$	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8
Miles-Ezzell	9.79%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
Modigliani-Miller	9.79%	9.82%	9.86%	9.90%	9.94%	9.99%	10.04%	10.09%
Fernández (2004)	9.79%	9.81%	9.83%	9.85%	9.87%	9.89%	9.92%	9.94%

**Table VIII**

**$K_u$  as a function of  $g$  (growth) and  $\alpha$  (required return to the increase of assets) if the required return to the profit after tax of the unlevered company ( $K_{PAT_u}$ ) is fixed**

$K_{PAT_u} = 9\%$ ;  $FCF_0 = 50$ ;  $D_0 = 375$ ;  $R_F = 5\%$ ;  $T = 40\%$ .

		<b>g</b>					
		0%	1%	2%	3%	4%	5%
<b><math>\alpha</math></b>	7%	9.00%	9.57%	10.30%	11.47%	14.26%	89.00%
	8%	9.00%	9.23%	9.49%	9.79%	10.19%	10.86%
	9%	9.00%	9.00%	9.00%	9.00%	9.00%	9.00%
	10%	9.00%	8.83%	8.67%	8.54%	8.43%	8.36%
	12%	9.00%	8.59%	8.26%	8.02%	7.88%	7.84%
	15%	9.00%	8.37%	7.93%	7.65%	7.52%	7.55%

Appendix 1  
**Derivation of formulas for Miles-Ezzell:  $D_t = K E_t$**

We are valuing a company with no leverage cost. The cost of debt is the risk-free rate ( $R_F$ ). The company is a growing perpetuity, which means that  $E_0[D_t] = D_0 (1+g)^t$ ,

**Value of tax shields**

The tax shield of the next period ( $t = 1$ ) is known with certainty ( $D_0 R_F T$ ) and the appropriate discount rate is  $R_F$ . However, the appropriate discount rate<sup>11</sup> for the expected tax shields if  $t$  is bigger than 1 is  $Ku$ .<sup>12</sup>

$$VTS_0 = \frac{D_0 R_F T}{1 + R_F} + \frac{D_0(1+g)R_F T}{(1 + R_F)(1 + Ku)} + \frac{D_0(1+g)^2 R_F T}{(1 + R_F)(1 + Ku)^2} + \frac{D_0(1+g)^3 R_F T}{(1 + R_F)(1 + Ku)^3} + \dots$$

We have the sum of a geometric progression growing at a rate  $(1+g)/(1+Ku)$ , and

$$\boxed{VTS_0 = \frac{D_0 R_F T}{(Ku - g)} \frac{(1 + Ku)}{(1 + R_F)}} \quad (\text{A1.1})$$

**The appropriate discount rate for the expected increases of debt**

$$PV_0[\Delta D_1] = \frac{g D_0}{(1 + K_{\Delta D1})} = \frac{D_0(1+g)}{(1 + Ku)} - \frac{D_0}{(1 + R_F)}$$

$$\text{Some algebra permits to express } 1 + K_{\Delta D1} = \frac{g(1 + Ku)(1 + R_F)}{g(1 + R_F) + R_F - Ku} \quad (\text{A1.2})^{13}$$

Equation (A1.2) is asymptotic in  $g = (Ku - R_F)/(1 + R_F)$ , in our example in  $g = 3.846\%$

g	-3%	-2%	0%	1%	2%	3%	4%	5%	6%	7%
$K_{\Delta D1} \text{ M-E}$	-52.0%	-62.5%	-100.0%	-138.8%	-220.5%	-503.9%	2189.0%	357.8%	198.6%	139.1%

<sup>11</sup> We define the “appropriate discount rate of a random variable”  $X$  as being the discount rate that, when applied to the expected value of the variable  $X$ , provides us with the present value of the variable:

$$PV_0[X_1] = \frac{E_0[X_1]}{(1 + \text{appropriate discount rate}_1)}$$

$$PV_0[X_2] = \frac{E_0[X_2]}{(1 + \text{appropriate discount rate}_1)(1 + \text{appropriate discount rate}_2)}$$

<sup>12</sup> Following equation (1):  $E_t + D_t = Vu_t + VTS_t$ , as  $D_t = K E_t$ , and because  $VTS_t$  is a function of  $D_t$ , it follows that  $D_t$ ,  $E_t$  and  $VTS_t$  have the same risk as  $Vu_t$ , and the same appropriate discount rate as  $Vu_t$ , that is  $Ku$ .

<sup>13</sup> Note that if  $g=0$ , then  $K_{\Delta D1}=-100\%$ . This does not make any economic sense because in this situation the expected value of the increase of debt is also 0.

## Appendix 1 (continued)

Now, for t=2:

$$PV_0[\Delta D_2] = \frac{D_0(1+g)^2}{(1+Ku)^2} - \frac{D_0(1+g)}{(1+R_F)(1+Ku)} = \frac{D_0[(1+g)^2 - (1+g)]}{(1+K_{\Delta D1})(1+K_{\Delta D2})} = \frac{g D_0 (1+g)}{(1+K_{\Delta D1})(1+K_{\Delta D2})}$$

As  $\frac{D_0(1+g)^2}{(1+Ku)^2} - \frac{D_0(1+g)}{(1+R_F)(1+Ku)} = \frac{g D_0}{(1+K_{\Delta D1})} \frac{(1+g)}{(1+Ku)}$ , it is obvious that  $K_{\Delta D2} = Ku$

The present value of all the expected increases of debt, using equation (12), is:

$PV_0[\Delta D_t] = VTS_0/T - D_0$ . Substituting VTS using equation (A1.1):

$$PV_0[\Delta D_t] = D_0 \frac{g(1+R_F) - Ku + R_F}{(Ku-g)(1+R_F)} \quad (A1.3)$$

Note that,  $PV_0[\Delta D_t] < 0$  if  $g < (Ku - R_F)/(1 + R_F)$ , in our example if  $g < 3.846\%$

### The appropriate discount rate for the expected equity cash flows

The relationship between expected values at t=1 of the free cash flow, the equity cash flow and the debt cash flow (CFd) is:

$E_0[ECF_1] = E_0[FCF_1] - E_0[CFd_1] + D_0 R_F T$ . This relationship is equivalent to:

$$E_0(Ke-g) = Vu_0(Ku-g) - D_0(R_F - g) + D_0 R_F T$$

As  $E_0 = Vu_0 - D_0 + VTS_0$ , then  $E_0 Ke = Vu_0 Ku - D_0 R_F + VTS_0 g + D_0 R_F T$

And the general equation for Ke is:

$$Ke = Ku + \frac{D_0}{E_0} [(Ku - R_F(1-T))] - \frac{VTS_0}{E_0} (Ku - g) \quad (A1.4)$$

Substituting (A1.1) in (A1.4):

$$Ke = Ku + \frac{D_0}{E_0} [(Ku - R_F(1-T))] - \frac{D_0 R_F T}{E_0} \frac{(1+Ku)}{(1+R_F)} = Ku + \frac{D_0}{E_0} (Ku - R_F) \left[ 1 - \frac{R_F T}{1+R_F} \right] \quad (A1.5)$$

But this expression is the average Ke. It is not the required return to equity ( $Ke_t$ ) for all periods. The value of the equity today is the sum of the present value of the equity cash flow of next period plus the present value of the equity value of next period:

$$E_0 = PV_0[ECF_1] + PV_0[E_1] = \frac{(1+g)ECF_0}{(1+Ke_1)} + \frac{(1+g)E_0}{(1+Ku)}$$

$$(1+Ke_1) = \frac{(1+g)ECF_0(1+Ku)}{E_0(Ku-g)} = \frac{E_0(Ke-g)(1+Ku)}{E_0(Ku-g)} = \frac{(Ke-g)(1+Ku)}{(Ku-g)} \quad (A1.6)$$

## Appendix 1 (continued)

Substituting (A1.5), we get:

$$\boxed{Ke_1 = Ku + \frac{D_0 (Ku - R_F)[1 + R_F(1 - T)](1 + Ku)}{E_0 (Ku - g)(1 + R_F)}} \quad (A1.7)$$

D <sub>0</sub>	0	100	200	300	375	400	500	600	700	800	900	999
Ke <sub>1</sub> M-E	9.00%	18.28%	29.78%	44.41%	58.30%	63.65%	90.06%	128.6%	190.1%	303.7%	584.7%	2368.1%

For t=2:

$$PV_0[ECF_2] = \frac{(1+g)^2 ECF_0}{(1+Ke_1)(1+Ke_2)} = \frac{(1+g)^2 FCF_0}{(1+Ku)^2} - \frac{D_0 R_F(1-T)(1+g)}{(1+R_F)(1+Ku)} + \frac{gD_0(1+g)}{(1 + \frac{gKu(1+R_F) - R_F + Ku}{g(1+R_F) + R_F - Ku})(1+Ku)}$$

Comparing this equation with the one for t=1, it is clear that  $Ke_2 = Ku$ .

### The appropriate discount rate for the expected taxes

If we assume that the appropriate discount rate for the expected increases of assets is also  $\alpha$ , then (see equation (7)), the present value of the expected taxes of the unlevered company is:

$$PV_0[Taxex_{U1}] = \frac{E_0[Taxex_{U1}]}{(1+K_{TAXU1})} = \left[ \frac{T}{1-T} \right] \left( \frac{(1+g)FCF_0}{(1+Ku)} + \frac{gA_0}{(1+\alpha)} \right)$$

As  $E[Taxes_{U1}] = [T/(1-T)] [FCF_0(1+g) + gA_0]$ , we can calculate  $K_{TAXU1}$ .

$$(1+K_{TAXU1}) = \frac{E_0[Taxex_{U1}]}{PV_0[Taxex_{U1}]} = \frac{(1+g)FCF_0 + gA_0}{(1+g)FCF_0(1+\alpha) + gA_0(1+Ku)} (1+Ku)(1+\alpha) \quad (A1.8)$$

If  $\alpha = Ku$ , then  $K_{TAXU1} = Ku$

$$\text{According to (9): } \boxed{Gu_0 = \frac{T}{1-T} \left[ Vu_0 + \frac{gA_0}{(\alpha - g)} \right]} \quad (A1.9)$$

To calculate the appropriate discount rate for the expected taxes of the levered company we use equation (8):

$$PV_0[Taxes_{L1}] = \frac{E_0[Taxes_{L1}]}{(1+K_{TAXL1})} = \left[ \frac{T}{1-T} \right] \left( \frac{(1+g)FCF_0}{(1+Ku)} + \frac{gA_0}{(1+\alpha)} - \frac{D_0 R_F(1-T)}{(1+R_F)} \right)$$

As  $E[Taxes_{L1}] = [T/(1-T)] [FCF_0(1+g) + gA_0 - D_0 R_F(1-T)]$

## Appendix 1 (continued)

$$1 + K_{TAXL1} = \frac{(1+g)FCF_0 + gA_0 - D_0R_F(1-T)}{\frac{(1+g)FCF_0}{(1+Ku)} + \frac{gA_0}{(1+\alpha)} - \frac{D_0R_F(1-T)}{(1+R_F)}} \quad (A1.10)$$

For  $t > 1$ , (for example, for  $t=2$ ), the present value is:

$$PV_0[Taxes_{L2}] = \frac{E_0[Taxes_{L1}](1+g)}{(1+K_{TAXL1})(1+K_{TAXL2})}$$

It is obvious that  $K_{TAXL2} = Ku$  if  $\alpha = Ku$

From equation (11) we can calculate the present value of the levered taxes:

$$G_{L0} = Gu_0 - VTS_0 = \frac{T}{1-T} \left[ Vu_0 + \frac{gA_0}{(\alpha-g)} \right] - \frac{D_0R_FT}{(Ku-g)} \frac{(1+Ku)}{(1+R_F)} \quad (A1.11)$$

Although  $K_{TAXU_t}$  and  $K_{TAXL_t}$  are not constant, we can calculate  $K_{TAXU}$  and  $K_{TAXL}$  such that  $G_{U0} = Taxes_{U0} (1+g) / (K_{TAXU} - g)$  and  $G_{L0} = Taxes_{L0} (1+g) / (K_{TAXL} - g)$ . Some algebra permits to find:

$$K_{TAXU} = \frac{Vu_0(\alpha-g)Ku + g\alpha A_0}{Vu_0(\alpha-g) + gA_0}$$

$$K_{TAXL} = g + \frac{E_0(Ke-g) + g(A_0 - D_0)}{Vu_0 + \frac{gA_0}{(\alpha-g)} - \frac{VTS_0(1-T)}{T}}$$

**The appropriate discount rate for the expected value of the unlevered equity (Vu)**

$$Vu_0 = \frac{FCF_0(1+g)}{(1+Ku)} + \frac{(1+g)Vu_0}{(1+K_{Vu1})}$$

$$(1+K_{Vu1}) = \frac{Vu_0(1+g)(1+Ku)}{Vu_0(1+Ku) - FCF_0(1+g)} = \frac{Vu_0(1+g)(1+Ku)}{Vu_0(1+Ku) - Vu_0(Ku-g)} = \frac{Vu_0(1+g)(1+Ku)}{Vu_0(1+g)} = 1+Ku$$

**The appropriate discount rate for the expected value of equity (E)**

$$E_0 = \frac{ECF_0(1+g)}{(1+Ke_1)} + \frac{E_0(1+g)}{(1+Ke_1)}$$

## Appendix 1 (continued)

$$(1 + K_{E1}) = \frac{E_0(1+g)}{E_0 - ECF_0(1+g)/(1+K_{e1})}. \text{ Using (A1.7) and knowing that } ECF_0(1-g) = E_0(Ke-g):$$

$$1 + K_{E1} = \frac{E_0(1+g)}{E_0 - \frac{E_0(Ke-g)(Ku-g)}{(Ke-g)(1+Ku)}} = \frac{(1+g)(1+Ku)}{(1+Ku) - (Ku-g)} = 1 + Ku.$$

$D_t = L \cdot E_t$  is absolutely equivalent to  $D_t = M \cdot Vu_t$ . In this case,  $\Delta D_t = X \cdot (FCF_t - FCF_{t-1}) = X \cdot \Delta FCF_t$ , where  $X = D_0 / FCF_0$ .

As, according to Miles Ezzel,  $Vu_t = E_t (1+L-L \cdot R_F \cdot T \cdot [(1+Ku)/(1+R_F)]/(Ku-g))$ , the relationship between L and M is:  $M = L / (1+L-L \cdot R_F \cdot T \cdot [(1+Ku)/(1+R_F)]/(Ku-g))$ . It is easy to find the relationship between M and X:  $D_0 = M \cdot Vu_0 = M \cdot FCF_0 (1+g)/(Ku-g) = M \cdot (D_0 / X)(1+g)/(Ku-g)$ , and  $X = M (1+g)/(Ku-g)$ .

## Appendix 2

**Derivation of formulas if debt is proportional to the book value of equity**

$D_t = K \cdot E_{bv_t}$ , where  $E_{bv}$  is the book value of equity. Then  $\Delta D_t = K \cdot \Delta E_{bv_t}$  and the relationship between  $\Delta D_t$  and  $\Delta A_t$  (increase of assets) is<sup>14</sup>  $\Delta D_t = \Delta A_t / (1+1/K)$ .

**The appropriate discount rate for the expected increases of debt**

As  $\Delta D_t = \Delta A_t / (1+1/K)$ , the appropriate discount rate for the increases of debt is the appropriate discount rate for the increases of assets. If the appropriate discount rate for the expected increases of assets is  $\alpha$ , then the present value of the increases of assets for a growing perpetuity is  $PV_0[\Delta D_t] = \frac{g D_0}{(\alpha - g)}$

**Value of tax shields**

In this situation, we can use equation (12),  $VTS_0 = T \cdot D_0 + T \cdot PV_0[\Delta D_t]$ , and the

value of the tax shields is: 
$$VTS_0 = \frac{D_0 \alpha T}{(\alpha - g)} \quad (A2.1)$$

If  $\alpha = Ku$ , then  $VTS_0 = T \cdot D_0 \cdot Ku / (Ku - g)$

**The appropriate discount rate for the expected value of the tax shields**

The tax shield of the next period ( $t=1$ ) is known with certainty ( $D_0 R_F T$ ) and the appropriate discount rate is  $R_F$ . The appropriate discount rate for the expected tax shield of  $t=2$  ( $K_{TS2}$ ) is:

$$PV_0[D_1 R_F T] = \frac{D_0(1+g)R_F T}{(1+R_F)(1+K_{TS2})} = R_F T \left[ \frac{D_0}{(1+R_F)^2} + \frac{gD_0}{(1+R_F)(1+\alpha)} \right]$$

$$K_{TS2} = \frac{R_F(1+\alpha) + g\alpha(1+R_F)}{(1+\alpha) + g(1+R_F)} \quad (A2.2)$$

If  $g=0$ ,  $K_{TS2} = R_F$

The present value of the tax shield of period  $t$  is:

$$PV_0[TS_t] = \frac{D_0(1+g)^{t-1} R_F T}{(1+R_F)(1+K_{TS2}) \dots (1+K_{TS_t})} =$$

$$= R_F T \left[ \frac{D_0}{(1+R_F)^t} + \frac{gD_0}{(1+R_F)^{t-1}(1+\alpha)} + \frac{gD_0(1+g)}{(1+R_F)^{t-2}(1+\alpha)^2} + \dots + \frac{gD_0(1+g)^{t-2}}{(1+R_F)(1+\alpha)^{t-1}} \right]$$

<sup>14</sup> The increase of the book value of equity is equal to the profit after tax (PAT) minus the equity cash flow. According to equation (5):  $\Delta E_{bv_t} = PAT_{L_t} - ECF_t = \Delta A_t - \Delta D_t = \Delta D_t / K$   
In this situation, the increase of debt is proportional to the increases of net assets, and the risk of the increases of debt is equal to the risk of the increases of assets:  $\Delta D_t = \Delta A_t / (1+1/K)$

## Appendix 2 (continued)

This expression is the sum of a geometric progression with a factor  $X = (1+g)(1+R_F)/(1+\alpha)$ . The solution is:  $PV_0[TS_t] = \frac{R_F T D_0}{(1+R_F)^{t-1}} \left[ \frac{1}{(1+R_F)} + \frac{g[X^{t-1}-1]}{(1+\alpha)[X-1]} \right]$  (A2.3)

$$(1+K_{TS_t}) = \frac{PV_0[TS_{t-1}]}{PV_0[TS_t]}(1+g) = X(1+\alpha) \frac{(1+\alpha)(X-1) + g(1+R_F)(X^{t-2}-1)}{(1+\alpha)(X-1) + g(1+R_F)(X^{t-1}-1)}$$

When  $t$  tends to infinity,  $K_{TS2} = \text{MIN}[\alpha, (1+R_F)(1+g)-1]$

**The appropriate discount rate for the expected equity cash flows**

Substituting (A2.1) in (A1.4):

$$Ke = Ku + \frac{D_0}{E_0} \left[ (Ku - R_F(1-T)) - \frac{\alpha T(Ku - g)}{(\alpha - g)} \right] \quad (A2.4)$$

Note that if  $\alpha = Ku$ :  $Ke = Ku + \frac{D_0}{E_0}(1-T)(Ku - R_F)$

But this expression is the average  $Ke$ . It is not the required return to equity ( $Ke_t$ ) for all the periods. But  $Ke_t$  is not constant for  $t$ . For  $t = 1$ , the relationship among the cash flows is:  $ECF_1 = FCF_1 - D_0 R_F (1-T) + \Delta D_1$

Calculating the expected value at  $t=0$  for a growing perpetuity and substituting the value of  $K_{\Delta D} = \alpha$ :

$$PV_0[ECF_1] = \frac{(1+g)ECF_0}{(1+Ke_1)} = \frac{(1+g)FCF_0}{(1+Ku)} - \frac{D_0 R_F(1-T)}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$\frac{E_0(Ke - g)}{(1+Ke_1)} = \frac{Vu_0(Ku - g)}{(1+Ku)} - \frac{D_0 R_F(1-T)}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$(1+Ke_1) = \frac{E_0(Ke - g)}{Vu_0 \frac{(Ku - g)}{(1+Ku)} + D_0 \left[ \frac{g}{(1+\alpha)} - \frac{R_F(1-T)}{(1+R_F)} \right]} \quad (A2.5)$$

$$\text{if } \alpha = Ku: \quad (1+Ke_1) = \frac{(Ke - g)(1+R_F)(1+Ku)}{(Ku - g)(1+R_F) + (Ke - Ku)} \quad (A2.6)$$

It is obvious that:

$$(1+Ke_t) = \frac{PV_0[ECF_{t-1}]}{PV_0[ECF_t]}(1+g)$$



## Appendix 2 (continued)

For  $t > 2$

$$PV_0[ECF_t] = \frac{(1+g)^t FCF_0}{(1+Ku)^t} - \frac{D_0(1+g)^{t-1} R_F(1-T)}{(1+R_F)(1+K_{TS2}) \dots (1+K_{TS t})} + \frac{gD_0(1+g)^{t-1}}{(1+\alpha)^t}$$

Using (A2.3), if we define  $X = (1+g)(1+R_F)/(1+\alpha)$ :

$$PV_0[ECF_t] = \frac{(1+g)^t FCF_0}{(1+Ku)^t} - \frac{D_0 R_F(1-T)}{(1+R_F)^{t-1}} \left[ \frac{1}{(1+R_F)} + \frac{g(X^{t-1}-1)}{(1+Ku)(X-1)} \right] + \frac{gD_0(1+g)^{t-1}}{(1+\alpha)^t}$$

Note that  $PV_t[ECF_t] < 0$  means only that  $PV_t[FCF_t + \Delta D_t] < PV_t[D_{t-1} R_F(1-T)]$

### The appropriate discount rate for the expected taxes

(A1.9) and (A1.10) also apply to this situation

From equation (11) we can calculate the present value of the levered taxes:

$$G_{L0} = Gu_0 - VTS_0 = \frac{T}{1-T} \left[ Vu_0 + \frac{gA_0 - D_0\alpha(1-T)}{(\alpha-g)} \right] \quad (A2.7)$$

### The appropriate discount rate for the expected value of tax shields (VTS)

$$VTS_0 = PV_0[TS_1] + PV_0[VTS_1] = \frac{D_0 T R_F}{(1+R_F)} + \frac{VTS_0(1+g)}{(1+K_{VTS1})};$$

$$(1+K_{VTS1}) = \frac{\alpha(1+R_F)(1+g)}{(\alpha+gR_F)}$$

For  $t = 2$ .

$$VTS_0 = PV_0[TS_1] + PV_0[TS_2] + PV_0[VTS_2] = PV_0[TS_1] + PV_0[TS_2] + \frac{VTS_0(1+g)^2}{(1+K_{VTS1})(1+K_{VTS2})}$$

$$(1+K_{VTS2}) = \frac{VTS_0(1+g)^2}{(1+K_{VTS1})(VTS_0 - PV_0[TS_1] - PV_0[TS_2])}$$

### The appropriate discount rate for the expected value of equity (E)

$$E_0 = \frac{ECF_0(1+g)}{(1+Ke_1)} + \frac{E_0(1+g)}{(1+Ke_1)}$$

## Appendix 2 (continued)

$$(1 + K_{E1}) = \frac{E_0(1+g)}{E_0 - ECF_0(1+g)/(1+K_{E1})}$$

For t =2.

$$E_0 = PV_0[ECF_1] + PV_0[ECF_2] + PV_0[E_2] = PV_0[ECF_1] + PV_0[ECF_2] + \frac{E_0(1+g)^2}{(1+K_{E1})(1+K_{E2})}$$

$$(1 + K_{E2}) = \frac{E_0(1+g)^2}{(1+K_{E1})(E_0 - PV_0[ECF_1] - PV_0[ECF_2])}$$

**The appropriate discount rate for the expected value of debt (D)**

$$\frac{E_0[D_1]}{(1+K_{D1})} = \frac{(1+g)D_0}{(1+K_{D1})} = \frac{D_0}{(1+R_F)} + \frac{gD_0}{(1+\alpha)}$$

$$(1+K_{D1}) = \frac{(1+g)(1+R_F)(1+\alpha)}{1+\alpha+g(1+R_F)}$$

Appendix 3  
**Derivation of formulas if  $D_t$  is known with certainty at  $t=0$ .**

This is the Modigliani-Miller assumption. This situation can be analyzed as a special case of Appendix 2: that of a company with  $\alpha = R_F$ . The value of the tax shield is

$$VTS_0 = \frac{D_0 R_F T}{(R_F - g)} \quad (\text{A3.1})$$

**The appropriate discount rate for the expected value of the tax shields is  $R_F$**

**The appropriate discount rate for the expected value of the equity cash flow**

Substituting (A3.1) in (A1.4), or substituting  $\alpha$  by  $R_F$  in (A2.4):

$$K_e = K_u + \frac{D_0}{E_0} \left[ (K_u - R_F(1-T)) - R_F T \frac{(K_u - g)}{(R_F - g)} \right] \quad (\text{A3.2})$$

But this expression is the average  $K_e$ . It is not the required return to equity ( $K_{e_t}$ ) for all the periods. Substituting  $\alpha$  by  $R_F$  in (A2.5):

$$(1 + K_{e_1}) = \frac{E_0(K_e - g)(1 + K_u)(1 + R_F)}{V_{u_0}(K_u - g)(1 + R_F) + D_0(1 + K_u)[g - R_F(1-T)]} \quad (\text{A3.3})$$

$$\text{It may be expressed also as: } (1 + K_{e_1}) = \frac{(K_e - g)(1 + K_u)(1 + R_F)}{(K_u - g)(1 + R_F) + (K_e - K_u)(1 + g)}$$

$$\text{For } t=2, \text{ it is obvious that: } PV_0[ECF_2] = PV_0[ECF_1] \frac{(1 + g)}{(1 + K_{e_2})}$$

**The appropriate discount rate for the expected taxes**

(A1.9) and (A1.10) also apply to this situation.

$$\text{For } t = 2, \text{ the present value is: } PV_0[\text{Taxex}_{L2}] = \frac{E_0[\text{Taxex}_{L1}](1 + g)}{(1 + K_{\text{TAXL1}})(1 + K_{\text{TAXL2}})}$$

From equation (11) we can calculate the present value of the levered taxes:

$$G_{L0} = G_{u0} - VTS_0 = \frac{T}{1-T} \left[ V_{u0} + \frac{g A_0}{(\alpha - g)} \right] - \frac{D_0 R_F T}{(R_F - g)} \quad (\text{A3.4})$$

## Appendix 3 (continued)

**The appropriate discount rate for the expected value of tax shields (VTS)**

Another way of finding the same result:

$$VTS_0 = PV_0[TS_1] + PV_0[VTS_1].$$

$$VTS_0 = \frac{D_0 TR_F}{(1 + R_F)} + \frac{VTS_0 (1 + g)}{(1 + K_{VTS1})}$$

$$\frac{D_0 R_F T}{(R_F - g)} = \frac{D_0 TR_F}{(1 + R_F)} + \frac{D_0 R_F T}{(R_F - g)} \frac{(1 + g)}{(1 + K_{VTS1})}$$

$$\frac{(1 + g)}{(R_F - g)(1 + K_{VTS1})} = \frac{(1 + R_F) - (R_F - g)}{(R_F - g)(1 + R_F)} = \frac{(1 + g)}{(R_F - g)(1 + R_F)}.$$

It is obvious that  $K_{VTS1} = R_F = K_{VTS_t}$

**The appropriate discount rate for the expected value of equity (E)**

Calculating the present value of equation (1) at  $t = 1$ :

$$PV_0[VTS_1] = PV_0[E_1] + PV_0[D_1] - PV_0[Vu_1].$$

$$VTS_0 \frac{(1 + g)}{(1 + R_F)} = E_0 \frac{(1 + g)}{(1 + K_{E1})} + D_0 \frac{(1 + g)}{(1 + R_F)} - Vu_0 \frac{(1 + g)}{(1 + Ku)}$$

$$\frac{E_0}{(1 + K_{E1})} = \frac{VTS_0 - D_0}{(1 + R_F)} + \frac{Vu_0}{(1 + Ku)} = \frac{(VTS_0 - D_0)(1 + Ku) + Vu_0(1 + R_F)}{(1 + R_F)(1 + Ku)} = \frac{E_0 + (VTS_0 - D_0)Ku + Vu_0 R_F}{(1 + R_F)(1 + Ku)}$$

$$(1 + K_{E1}) = \frac{E_0(1 + R_F)(1 + Ku)}{(E_0 - Vu_0)(1 + Ku) + Vu_0(1 + R_F)}$$

for  $t=2$

$$\frac{E_0}{(1 + K_{E1})(1 + K_{E2})} = \frac{E_0 - Vu_0}{(1 + R_F)^2} + \frac{Vu_0}{(1 + Ku)^2} = \frac{(E_0 - Vu_0)(1 + Ku)^2 + Vu_0(1 + R_F)^2}{(1 + R_F)^2(1 + Ku)^2}$$

$$(1 + K_{E1})(1 + K_{E2}) = \frac{E_0(1 + R_F)^2(1 + Ku)^2}{(E_0 - Vu_0)(1 + Ku)^2 + Vu_0(1 + R_F)^2}$$

$$(1 + K_{Et}) = (1 + R_F)(1 + Ku) \frac{(E_0 - Vu_0)(1 + Ku)^{t-1} + Vu_0(1 + R_F)^{t-1}}{(E_0 - Vu_0)(1 + Ku)^t + Vu_0(1 + R_F)^t}$$

**The appropriate discount rate for the expected value of debt (D)**

$$\frac{E_0[D_1]}{(1+K_{D1})} = \frac{(1+g)D_0}{(1+K_{D1})} = \frac{D_0}{(1+R_F)} + \frac{gD_0}{(1+R_F)} = \frac{(1+g)D_0}{(1+R_F)}$$

Then,  $K_{Dt} = R_F$

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